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A METRIC CHARACTERIZATION OF ZERO-DIMENSIONAL SPACES

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ABSTRACT. It is shown that a nonempty separable metrizable space X is zero-dimensional if and only if there exists a metric ρ on X, inducing the given topology of X and such that all nonzero distances $\rho(x, y)$ are mutually different.

1. **Introduction.** Sometimes it is possible to characterize topological properties of a metrizable space X by claiming that a metric having certain properties can be introduced on X. J. de Groot [**1**] gave a characterization of a metrizable separable space of dim $\leqq n$ by means of a totally bounded metric satisfying certain inequalities. Similar results were obtained by J. Nagata [**2**]. The purpose of this note is to show that the metric which we call strongly rigid characterizes zero-dimensionality.

DEFINITION 1.1. A metric space (X, ρ) is said to be *strongly rigid* if all nonzero distances $\rho(x, y)$ are mutually different, which means that $\rho(x, y) = \rho(u, v)$ and $x \neq y$ imply that $\{x, y\} = \{u, v\}$.

REMARK 1.1. We are using here the modifier "strongly" since under "rigid metric space" is understood a metric space having no nontrivial isometry.

DEFINITION 1.2. A metrizable space X is said to be *eventually strongly rigid* if there is a strongly rigid metric on X inducing the topology of X.

REMARK 1.2. It is evident that any subset $Y \subset X$ of an eventually strongly rigid space X is again eventually strongly rigid with respect to its relative topology.

THEOREM. *A nonempty separable metrizable space X is zero-dimensional if and only if it is eventually strongly rigid.*

We accomplish the proof of this statement showing that each point in a strongly rigid space has arbitrarily small spherical neighborhoods with empty boundary, and that the Cantor set is eventually strongly rigid.

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2. Proof of the Theorem.

DEFINITION 2.1. Let (X, ρ) be a metric space, $r > 0$ and $x \in X$, we denote by $S(x, r)$ the r-sphere about x: $S(x, r) = \{y \mid y \in X \text{ and } \rho(x, y) = r\}$. It is obvious that, if $S(x, r)$ is empty, then the boundary of the r-ball about x is also empty.

LEMMA 2.1. *If (X, ρ) is a strongly rigid metric space, then for each $x \in X$ and each $\varepsilon > 0$ there exists $r \in (0, 2\varepsilon)$ such that $S(x, r)$ is empty.*

PROOF. First we observe that each sphere in a strongly rigid space contains no more than one point. If $S(x, \varepsilon)$ is empty, we are done. If not, there is a point, say $y \in S(x, \varepsilon)$. If $S(x, \varepsilon/2)$ is empty, we are done again; if not, there is a point, say $z \in S(x, \varepsilon/2)$, and we observe that $\varepsilon/2 < \rho(y, z) \leqq 2\varepsilon$. Putting $r = \rho(y, z)$, we conclude that $S(x, r)$ must be empty, since otherwise the distance from x to some point would be the same as $\rho(y, z)$ which is impossible, and this accomplishes our proof.

LEMMA 2.2. *The Cantor set C is eventually strongly rigid.*

PROOF. We represent the Cantor set C in the classical form: $C = \bigcap_{n=1}^{\infty} A^n$ where $A^1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3})$, $A^2 = A^1 \setminus [(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})]$ and so on. The components of A^n we denote by $C_1^n, C_2^n, \cdots, C_{2^n}^n$.

Let now $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive numbers $a_n > 0$, having the property that for each $n = 1, 2, \cdots$ we have $a_n > \sum_{k=n+1}^{\infty} a_k$. Such series exist; for example, the geometrical series $a_n = 3^{-n}$ has this property. Now we observe the following property of our series $\sum a_n$ which will play the crucial role in the construction of the strongly rigid metric ρ on C: If $k(1) < k(2) < k(3) < \cdots$ and $l(1) < l(2) < l(3) < \cdots$ are two different sequences of natural numbers, then the subseries $\sum_{n=1}^{\infty} a_{k(n)}$ and $\sum_{n=1}^{\infty} a_{l(n)}$ have different values. Arranging our series $\sum a_n$ in the scheme:

$$\begin{array}{l} a_1^1 \\ a_1^2, a_2^2, a_3^2 \\ a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3 \\ a_1^n, a_2^n, \cdots, a_{2^n-1}^n \end{array}$$

where $a_1^1 = a_1$, $a_1^2 = a_2$, $a_2^2 = a_3 \cdots$ and so on, we define the expressions $\rho^n(x, y)$ for $n = 1, 2, \cdots$ and $x, y \in C$ in the following way:

Let $x, y \in C$. If x and y are in the same component of A^n, we put $\rho^n(x, y) = 0$, and if $x \in C_k^n$ and $y \in C_l^n$ (assuming for example $x < y$) we put $\rho^n(x, y) = a_k^n + a_{k+1}^n + \cdots + a_{l-1}^n$. Defining finally $\rho(x, y)$ by: $\rho(x, y) = \sum_{n=1}^{\infty} \rho^n(x, y)$ we see that $\rho(x, y)$ can be represented as a certain subseries of $\sum a_n$ and that $\rho(x, y)$ is a metric on C, since its symmetry and the triangle inequality follow directly from the definition and if $x \neq y$ there is evidently an index n such that $\rho^n(x, y) > 0$. Moreover, the metric $\rho(x, y)$ is topologically equivalent to $|x - y|$ since if $\{x_k\}$, $x \in C$ and $|x_k - x| \to 0$, then the minimal index n for which x_k and x belong to different components of A^n tends to ∞ as $k \to \infty$ and therefore the first member a_i^n appearing in the expression for $\rho(x_k, x)$ tends to zero, thus the expression $\rho(x_k, x)$ itself tends to zero. Conversely, if $\rho(x_k, x) \to 0$ and if $|x_k - x|$ were not converging to zero, then due to compactness of C there would be a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ converging to some $y \neq x$, but then, using the above argument, it would follow that also $\rho(x_{k(n)}, y) \to 0$, which is impossible. It remains to show that if $\{x, y\}$ and $\{u, v\}$ are two different pairs of distinct points in C, then $\rho(x, y) \neq \rho(u, v)$. It is evident that there exists some n such that $\rho^n(x, y) \neq \rho^n(u, v)$ (assuming for example $x \neq u$, $x < u$, it suffices to choose n such that x and u are in different components of A^n). But this implies that $\rho(x, y)$ and $\rho(u, v)$ are represented by different subseries of $\sum a_n$ and therefore have different values. Hence, $\rho(x, y)$ is strongly rigid on C, and C is eventually strongly rigid, which proves our lemma.

Now we have all we needed to prove our theorem. If X is a nonempty eventually strongly rigid space, then from Lemma 2.1 follows that $\dim X = 0$. If on the other hand X is separable, metrizable, and zero-dimensional, it is known that X can be topologically embedded in the Cantor set C and from Lemma 2.2 and Remark 2.1 follows that X is eventually strongly rigid.

References

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