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GROUP RINGS SATISFYING A POLYNOMIAL IDENTITY. III

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ABSTRACT. Let $K[G]$ denote the group ring of G over the field K and let Δ denote the F.C. subgroup of G. In this paper we show that if $K[G]$ satisfies a polynomial identity of degree n, then $[G:\Delta] \leqq n/2$. Moreover this bound is best possible.

If $K[G]$ satisfies a polynomial identity of degree n, then it is known that $[G:\Delta] < \infty$. In fact if $K[G]$ is prime or if K has characteristic 0 then $[G:\Delta] \leqq (n/2)^2$ by the results of **[4]**. In general we have $[G:\Delta] \leqq n!$ by the results of **[1]**. Thus the goal of this paper is to sharpen these to obtain the best possible bound, namely $[G:\Delta] \leqq n/2$. We follow the notation of **[3]**.

1. **The abelian case.** Throughout this section we assume that $[G:\Delta] < \infty$ and that Δ is abelian. Let $x_1 = 1, x_2, x_3, \cdots, x_m$ be a complete set of $m = [G:\Delta]$ coset representatives for Δ in G.

LEMMA 1.1. *There exists a K-monomorphism $\rho: K[G] \to K[\Delta]_m$, where the latter is the ring of $m \times m$ matrices over $K[\Delta]$, satisfying*
(i) *for $a \in \Delta$, $\rho(a) = \operatorname{diag}(a^{x_1}, a^{x_2}, \cdots, a^{x_m})$,*
(ii) $\rho(x_i)e_{11} = e_{i1}$, $e_{11}\rho(x_i^{-1}) = e_{1i}$,
where $\{e_{ij}\}$ is the set of matrix units in $K[\Delta]_m$.

PROOF. Since Δ is normal in G, $\{x_1^{-1}, x_2^{-1}, \cdots, x_m^{-1}\}$ is also a complete set of coset representatives for Δ in G. Set $V = K[G]$. Then clearly V is a left $K[\Delta]$-module with free basis $\{x_1^{-1}, x_2^{-1}, \cdots, x_m^{-1}\}$. Now V is also a right $K[G]$-module and as such it is faithful. Since right and left multiplication commute as operators on V, it follows that $K[G]$ is a set of $K[\Delta]$-linear transformations on an m-dimensional free $K[\Delta]$-module V. Thus there exists a K-monomorphism ρ with $\rho(K[G]) \subseteq K[\Delta]_m$.

Let $a \in \Delta$. Then $x_i^{-1}a = (x_i^{-1}ax_i)x_i^{-1} = a^{x_i}x_i^{-1}$; so clearly $\rho(a) = \operatorname{diag}(a^{x_1}, a^{x_2}, \cdots, a^{x_m})$.

Now to compute $e_{11}\rho(x_i^{-1})$ we need only consider the first row of the matrix $\rho(x_i^{-1})$. Since $x_1x_i^{-1} = x_i^{-1}$ we see that this first row is precisely e_{1i}; so $e_{11}\rho(x_i^{-1}) = e_{11}e_{1i} = e_{1i}$.

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Finally to compute $\rho(x_i)e_{11}$ we need only look at the first column of the matrix $\rho(x_i)$. Since $x_j^{-1}x_i \notin \Delta$ for $j \neq i$ and since $x_j^{-1}x_i = 1 = x_1$ for $j = i$ we see that this first column is precisely e_{i1}. Thus $\rho(x_i)e_{11} = e_{i1}e_{11} = e_{i1}$ and the result follows.

Let $K[\Delta]$ be embedded naturally in $K[\Delta]_m$ as the set of scalar matrices. Since Δ is abelian this is a central subring of $K[\Delta]_m$. Let $R = K[\Delta] \cdot \rho(K[G])$ be the subring of $K[\Delta]_m$ generated by $K[\Delta]$ and $\rho(K[G])$. We will show below that R is in some sense a large subring of $K[\Delta]_m$.

LEMMA 1.2. *For each $i = 2, 3, \cdots, m$, set*

$$H_i = \{(a, x_i) = a^{-1}x_i^{-1}ax_i \mid a \in \Delta\}.$$

Then H_i is an infinite subgroup of Δ.

PROOF. Since Δ is a normal abelian subgroup of G, the map $\eta_i : \Delta \to \Delta$ given by $a \to a^{-1}a^{x_i}$ is an endomorphism. Clearly H_i is the image of η_i so H_i is a subgroup of Δ and $C_\Delta(x_i)$ is the kernel of η_i. Thus $[\Delta : C_\Delta(x_i)] = |H_i|$. If $|H_i| < \infty$, then $[\Delta : C_\Delta(x_i)] < \infty$ and since $[G:\Delta] < \infty$ we would have $[G : C_G(x_i)] < \infty$ and $x_i \in \Delta$, a contradiction. Thus H_i is infinite.

For each $i = 2, 3, \cdots, m$, let S_i be the augmentation ideal of $K[H_i]$. Thus

$$S_i = \{\textstyle\sum k_g g \in K[H_i] \mid \sum k_g = 0\}.$$

Then S_i is a K-algebra (without 1) which has as a K-basis the elements $1 - g$ with $g \in H_i$, $g \neq 1$.

Now $S_i \subseteq K[\Delta]$ and $K[\Delta]$ is commutative. We define $S = S_2S_3 \cdots S_m$ to be the set of all finite K-linear sums of products $s_2s_3 \cdots s_m$ with $s_i \in S_i$. Since $K[\Delta]$ is commutative, S is a K-subalgebra (without 1) of $K[\Delta]$.

LEMMA 1.3. *S is not a nilpotent ring.*

PROOF. It clearly suffices to show that for each $i = 2, 3, \cdots, m$ and $\alpha \in K[\Delta]$ that $S_i\alpha = 0$ implies $\alpha = 0$. Suppose $S_i\alpha = 0$ and let $g \in H_i$. Then $1 - g \in S_i$ so $(1 - g)\alpha = 0$. Thus $\alpha = g\alpha$ and $(\operatorname{Supp} \alpha) = g(\operatorname{Supp} \alpha)$. Therefore H_i permutes by left multiplication the finite set $\operatorname{Supp} \alpha \subseteq \Delta$. If $\alpha \neq 0$ then $\operatorname{Supp} \alpha \neq \varnothing$ and this would imply easily that H_i is finite, a contradiction by Lemma 1.2. Thus $\alpha = 0$ and the result follows.

LEMMA 1.4. *With the above notation we have $R \supseteq (S)_m$, the ring of $m \times m$ matrices over S.*

PROOF. Recall that $K[\Delta]$ is contained in $K[\Delta]_m$ as scalar matrices and that $R = K[\Delta] \cdot \rho(K[G])$. Let $i = 2, 3, \cdots$, or m and let $a \in \Delta$. Then

$$a^{-1}(\rho(a) - a^{x_i}) \in R.$$

The above matrix is diagonal and we will consider the 1st and ith entries. The ith entry is $a^{-1}(a^{x_i} - a^{x_i}) = 0$ by Lemma 1.1(i) and the 1st entry is

$$a^{-1}(a - a^{x_i}) = 1 - a^{-1}a^{x_i} = 1 - (a, x_i).$$

Thus for any element $g \in H_i$, R contains a matrix of the form

$$\operatorname{diag}(1 - g, *, 0, *)$$

where the 0 is in the ith position. Since R is a K-algebra and since every element of S_i is a K-linear sum of such terms $1 - g$ we see that for each $s_i \in S_i$, R contains a matrix of the form $\alpha_i = \operatorname{diag}(s_i, *, 0, *)$.

Now choose $s_i \in S_i$ for $i = 2, 3, \cdots, m$ and let α_i be as above. Then $\alpha = \alpha_2\alpha_3 \cdots \alpha_m \in R$ and

$$\alpha = \operatorname{diag}(s_2s_3 \cdots s_m, 0, 0, \cdots, 0) = s_2s_3 \cdots s_m e_{11}$$

where $\{e_{jk}\}$ is the usual set of matrix units. This clearly implies that $R \supseteq Se_{11}$.

Finally let e_{jk} be any matrix unit. Then, by Lemma 1.1(ii), $R \supseteq \rho(x_j)(Se_{11})\rho(x_k^{-1}) = Se_{jk}$ and $R \supseteq (S)_m$.

PROPOSITION 1.5. *Let $K[G]$ satisfy a polynomial identity of degree n and suppose further that $[G:\Delta] < \infty$ and Δ is abelian. Then $[G:\Delta] \leqq n/2$.*

PROOF. By Lemma 5.3 of [**1**], $K[G]$ satisfies an identity of the form

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \zeta_1\zeta_2 \cdots \zeta_n + \sum_{\sigma \in \mathrm{Sym}_n; \sigma \neq 1} k_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}.$$

Then of course $\rho(K[G])$ also satisfies f. Since f is multilinear and $K[\Delta]$ is central in $K[\Delta]_m$, it then follows easily that $R = K[\Delta] \cdot \rho(K[G])$ satisfies f. By Lemma 1.4, $R \supseteq (S)_m$, so $(S)_m$ also satisfies f.

Suppose by way of contradiction that $m = [G:\Delta] > n/2$. Since S is not nilpotent by Lemma 1.3 we can choose $s^{(1)}, s^{(2)}, \cdots, s^{(n)} \in S$ with $s^{(1)}s^{(2)} \cdots s^{(n)} \neq 0$. Since $n < 2m$ we may set $\zeta_1 = s^{(1)}e_{11}$, $\zeta_2 = s^{(2)}e_{12}$, $\zeta_3 = s^{(3)}e_{22}$, $\zeta_4 = s^{(4)}e_{23}$, $\zeta_5 = s^{(5)}e_{33}, \cdots$. Then $\zeta_1\zeta_2 \cdots \zeta_n$ evaluated at these values is $s^{(1)}s^{(2)} \cdots s^{(n)}e_{1j} \neq 0$ where $j = [n/2] + 1$. On the other hand for all $\sigma \in \mathrm{Sym}_n$, $\sigma \neq 1$, $\zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$ evaluated at these values is zero. Thus $(S)_m$ does not satisfy f, a contradiction. Therefore $m \leqq n/2$ and the result follows.

2. **The general case.**

Let $\Delta_k(G)$ be defined as in [**3**].

LEMMA 2.1. *Suppose there exists an integer k with $[G:\Delta_k(G)] < \infty$. Then $[G:\Delta] < \infty$ and $|\Delta'| < \infty$.*

PROOF. Since $\Delta \supseteq \Delta_k$ and $[G:\Delta_k] < \infty$ we have $[G:\Delta] < \infty$. Now Δ is a subgroup of G so every right translate of Δ_k in G is either entirely contained in Δ or is disjoint from it. This implies that $[\Delta:\Delta_k] < \infty$ and say $\Delta = \Delta_k y_1 \cup \Delta_k y_2 \cup \cdots \cup \Delta_k y_r$.

Since each $y_i \in \Delta$ we can set $u = \max_i [G:C(y_i)] < \infty$. If $x \in \Delta$ then $x \in \Delta_k y_i$ for some i and this implies easily that $[G:C(x)] \leqq uk$. Thus $[\Delta:C_\Delta(x)] \leqq uk$ and by Theorem 4.4(ii) of [3], $|\Delta'| < \infty$.

We now come to the main result of this paper.

THEOREM 2.2. *Let $K[G]$ satisfy a polynomial identity of degree n. Then $[G:\Delta(G)] \leqq n/2$ and $|\Delta(G)'| < \infty$.*

PROOF. Set $k = (n!)^2$. Then by Theorem 3.4 of [3], $[G:\Delta_k(G)] < \infty$. Thus, by Lemma 2.1, $[G:\Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$. Set $H = \Delta(G)'$ and consider $\bar{G} = G/H$. If $x \in \Delta(G)$ then clearly $\bar{x}$, its image in $\bar{G}$, has only finitely many conjugates and $\bar{x} \in \Delta(\bar{G})$. Conversely suppose $\bar{x} \in \Delta(\bar{G})$. Then conjugates of x are contained in only finitely many cosets of H. Since H is finite, x has only finitely many conjugates and $x \in \Delta(G)$. Thus $\Delta(\bar{G}) = \Delta(G)/H$.

Consider $K[\bar{G}]$. Since $K[\bar{G}]$ is an epimorphic image of $K[G]$ we see that $K[\bar{G}]$ satisfies a polynomial identity of degree n. Since $\Delta(\bar{G}) = \Delta(G)/H$ and $H = \Delta(G)'$ we see that $\Delta(\bar{G})$ is abelian and $[\bar{G}:\Delta(\bar{G})] < \infty$. By Proposition 1.5 we have finally $[G:\Delta(G)] = [\bar{G}:\Delta(\bar{G})] \leqq n/2$ and the result follows.

The following corollary shows that the above bound $n/2$ is best possible. The result is an immediate consequence of Theorems 1.1(i) and 1.3(i) of [3] and Theorem 2.2.

COROLLARY 2.3. *Let n be a positive integer and suppose that G is a group with $\Delta(G)$ abelian. Then $[G:\Delta(G)] \leqq n/2$ if and only if $K[G]$ satisfies a polynomial identity of degree $\leqq n$.*

On the other hand, there is no fixed bound for the size of $\Delta(G)'$. For example, let A be a finite abelian group of odd order and let G be the extension of A by an element x of order 2 which acts in a dihedral manner on A (that is, $a^x = a^{-1}$ for all $a \in A$). Then G is finite so $G = \Delta(G)$ and $A = G'$ can be made arbitrarily large. Since G has an abelian subgroup of index 2, $K[G]$ satisfies a polynomial identity of degree 4 and this is independent of the size of $A = G'$.

Finally we remark that Theorem 2.2 answers in the affirmative Problem 4(i) of [2].

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