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SOME EXAMPLES RELATING THE DELETED PRODUCT TO IMBEDDABILITY[1]

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ABSTRACT. Examples are given showing the limitations of the homology of the deleted product in determining the imbeddability of simplicial complexes in a given Euclidean space. It is also proven that the only finite 1-complexes whose polyhedral deleted products are closed 2-manifolds are the two primitive skew curves of Kuratowski.

1. **Introduction.** The deleted product of a topological space X is defined to be the space

$$X^* = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2\}.$$

In [7] we modified the techniques developed by C. W. Patty in [5] to prove the following theorem which is a version of Theorem 2 of [6]:

THEOREM 1.1. *A collapsible 2-dimensional simplicial complex X can be imbedded in Euclidean 2-space if and only if $H_2(X^*) = 0$ and $H_3(X^*) = 0$, where $H_*(\ )$ denotes singular homology with integer coefficients.*

In this article we present examples which show that the homology of the deleted product fails to distinguish imbeddability if either a larger class of spaces X is considered or if imbeddability in 2-space is weakened to imbeddibility in 3-space. Specifically, we prove:

PROPOSITION 1.2. *There are finite 1-dimensional simplicial complexes A_1 and A_2 such that $H_*(A_1^*) \cong H_*(A_2^*)$, $H_*(A_1) \cong H_*(A_2)$, A_1 is planar (i.e., imbeddable in R^2) but A_2 is nonplanar.*

PROPOSITION 1.3. *There are collapsible 2-dimensional simplicial complexes B_1 and B_2 such that $H_*(B_1^*) \cong H_*(B_2^*)$, B_1 can be imbedded in R^3, but B_2 cannot be imbedded in R^3.*

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We also show that Theorem 1.1 is the best possible result in the following sense:

PROPOSITION 1.4. *There are nonplanar collapsible 2-dimensional simplicial complexes C_1 and C_2 such that $H_2(C_1^*) \neq 0$, $H_3(C_1^*) = 0$, $H_2(C_2^*) = 0$, and $H_3(C_2) \neq 0$.*

The complexes A_2, B_2, and C_2 are constructed from the 1-dimensional complexes K_5 (the complete graph on five vertices) and $K_{3,3}$ (the complete bipartite graph on two sets of three vertices). It is proven in [2] that $P(K_5^*)$ is a 2-sphere with six handles while $P(K_{3,3}^*)$ is a 2-sphere with four handles, where $P(X^*) = \{(x_1, x_2) \in X^* \mid x_1$ and x_2 lie in disjoint closed simplices of $X\}$ is the polyhedral deleted product of the finite complex X and is a deformation retract of X^* (cf. [3]). We conclude the paper with the following theorem which parallels Kuratowski's theorem (that a finite 1-complex is nonplanar if and only if it has a subcomplex homeomorphic to either K_5 or $K_{3,3}$):

THEOREM 1.5. *If X is a finite 1-dimensional simplicial complex, then $P(X^*)$ is a closed 2-manifold if and only if X is K_5 or $K_{3,3}$.*

2. **Computational lemmas.** In [4], C. W. Patty attempted to provide a stepwise algorithm for computing the homology of the deleted product of an arbitrary finite 1-dimensional complex. Unfortunately, two crucial parts (Theorems 4.1 and 4.2) of his algorithm are incorrect, as can be seen by considering the complex K_5; using Patty's algorithm one would conclude that the second homology of K_5^* has even rank and the first homology of K_5^* has odd rank. But since $P(K_5^*)$ is a 2-sphere with six handles, $H_2(K_5^*)$ has rank one and $H_1(K_5^*)$ has rank twelve.

For our purposes, it suffices to list three results which provide an algorithm for computing the homology of the deleted product of certain finite 1-complexes. The first is a restatement of Theorems 3.1 and 3.2 of [4], the second is a weakened (but valid) version of Theorems 4.1 and (4.2) of [4], and the third is a simple piecing together lemma. For proofs see [7]. Let Z^k denote the direct sum of k copies of the integers. The degree of a vertex v of a 1-complex X is the number of 1-simplices of X having v as a vertex.

PROPOSITION 2.1. *If X is the simplicial complex which is constructed by adding a 1-simplex to the 1-complex Y at a vertex of degree n in Y, then if Y is not an arc*

$$H_1(X^*) \cong H_1(Y^*) \oplus Z^{2n-2},$$
$$H_j(X^*) \cong H_j(Y^*) \quad \textit{if } j \neq 1.$$

PROPOSITION 2.2. *Suppose the simplicial complex X is constructed by adding a 1-simplex between the vertices v_1 and v_2 of the connected 1-complex Y, Y is not an arc. If every simple closed curve in $W = X - \bigcup_{i=1}^{2} \mathrm{st}\,(v_i, X)$ can be written as the sum (in the homology sense) of simple closed curves in W which do not separate v_1 and v_2 in Y, then*

$$H_1(X^*) \cong H_1(Y^*) \oplus H_0(W) \oplus H_0(W),$$
$$H_2(X^*) \cong H_2(Y^*) \oplus H_1(W) \oplus H_1(W),$$
$$H_j(X^*) \cong H_j(Y^*) \quad \text{if } j \neq 1, 2.$$

PROPOSITION 2.3. *Suppose the 1-complex X is the union of two connected subcomplexes X_1 and X_2 such that $X_1 \cap X_2$ is a 1-simplex S whose midpoint separates $X_1 - S$ and $X_2 - S$ in X. If neither X_1 nor X_2 is an arc, then*

$$H_1(X^*) \cong H_1(X_1^*) \oplus H_1(X_2^*) \oplus Z,$$
$$H_2(X^*) \cong H_2(X_1^*) \oplus H_2(X_2^*) \oplus H_2(X_1 \times X_2) \oplus H_2(X_1 \times X_2),$$
$$\tilde{H}_j(X^*) = 0 \quad \text{if } j \neq 1, 2.$$

The 2-dimensional complexes of our examples are cones over 1-complexes, and we will need the following homology version of a theorem of A. H. Copeland. Jr. (cf. [1]). Let CX denote the cone over X, which is taken to be the quotient space of $X \times I$ under the identification of $(x_1, 1)$ with $(x_2, 1)$ for all x_1 and x_2 in X. The equivalence class of CX containing (x, t) is denoted by $[x, t]$.

PROPOSITION 2.4. *If X is a finite simplicial complex, then there is an exact sequence*

$$\cdots \longrightarrow H_i(X^*) \xrightarrow{\phi_i} H_i(X) \oplus H_i(X)$$
$$\xrightarrow{\psi_i} H_i((CX)^*) \xrightarrow{\Delta} H_{i-1}(X^*) \longrightarrow \cdots$$

in which

$$\phi_i(u) = (p_{1_*}(u), p_{2_*}(u)), \qquad u \in H_i(X^*),$$
$$\psi_i(u_1, u_2) = g_{1_*}(u_1) - g_{2_*}(u_2), \qquad u_1, u_2 \in H_i(X),$$

where $p_i: X^* \to X$ is defined by $p_i(x_1, x_2) = x_i$ and $g_i: X \to (CX)^*$ is defined by $g_i(x) = ([x, 2 - i], [x, i - 1])$.

3. **The examples.** The 1-dimensional simplicial complexes A_1 and A_2 of Proposition 1.2 are drawn in Figure 3.1.

$H_*(A_1^*)$ is computed by starting with a simple closed curve and adding 1-simplices one at a time so that at each stage either Propostion 2.1 or 2.2 applies. Since A_2 consists of two copies of $K_{3,3}$ joined by a 1-simplex, $H_*(A_2^*)$ is easily computed using the remarks of §1 and Propositions 2.1

and 2.3. The result of these calculations is:

$$\tilde{H}_j(A_1^*) \cong \tilde{H}_j(A_2^*) = 0 \quad \text{for } j \neq 1, 2,$$
$$H_1(A_1^*) \cong H_1(A_2^*) \cong Z^{25},$$
$$H_2(A_1^*) \cong H_2(A_2^*) \cong Z^{34};$$

we also have

$$\tilde{H}_j(A_1) \cong \tilde{H}_j(A_2) = 0 \quad \text{for } j \neq 1, \quad H_1(A_1) \cong H_1(A_2) \cong Z^8.$$

Since A_1 is planar while A_2 is nonplanar, this proves Proposition 1.2.

Now set $B_1 = CA_1$ and $B_2 = CA_2$. Then B_1 and B_2 are collapsible 2-dimensional simplicial complexes, B_1 is imbeddable in R^3 while B_2 is not. Using Proposition 2.4 we have

$$\tilde{H}_j(B_1^*) \cong \tilde{H}_j(B_2^*) = 0 \quad \text{for } j \neq 2, 3,$$
$$H_2(B_1^*) \cong H_2(B_2^*) \cong Z^9,$$
$$H_3(B_1^*) \cong H_3(B_2^*) \cong Z^{34}$$

This proves Proposition 1.3.

Finally set $C_1 = CF$ where F is the disjoint union of a point and a simple closed curve, and let $C_2 = CK_5$. Then using Proposition 2.4 we have

$$H_2(C_1^*) \cong Z, \qquad H_2(C_2^*) = 0,$$
$$H_3(C_1^*) = 0, \qquad H_3(C_2^*) \cong Z.$$

Since both C_1 and C_2 are collapsible 2-dimensional nonplanar simplical complexes, this proves Proposition 1.4.

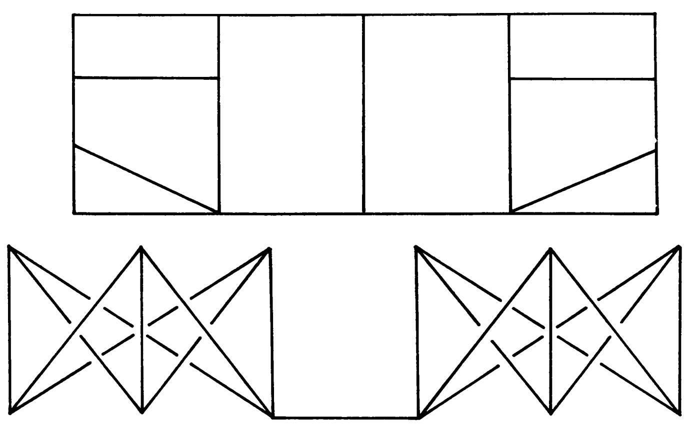

FIGURE 3.1. THE COMPLEXES A_1 (TOP) AND A_2.

4. **Proof of Theorem 1.5.** Suppose X is a finite 1-dimensional simplicial complex such that $P(X^*)$ is a closed 2-manifold. We will show that either $X = K_5$ or $X = K_{3,3}$.

First observe that X must be connected. For if A and B were distinct components of X, then $A \times B$ and $P(A^*)$ would be components of $P(X^*)$ and hence closed 2-manifolds. $A \times B$ could be a closed 2-manifold only if A is a simple close curve, but then $P(A^*)$ would not be a 2-manifold.

Since every 1-cell of $P(X^*)$ must be a face of exactly two 2-cells of $P(X^*)$ we can conclude:

(i) X contains no vertices of degree less than three or greater than four.

(ii) The closed star of a vertex of degree four in X contains both vertices of every 1-simplex of X.

(iii) The closed star of a vertex of degree three in X contains a vertex of every one simplex of X.

If X contains a vertex of degree four, then (ii) implies that X has exactly five vertices. So $X \subseteq K_5$ and hence $P(X^*) \subseteq P(K_5^*)$. Since $P(X^*)$ and $P(K_5^*)$ are closed manifolds we must have $P(X^*) = P(K_5^*)$ and hence $X = K_5$.

Finally, suppose every vertex of X has degree three. By the preceding case, X has more than five vertices. Choose a vertex u_0 of X and let u_1, u_2, and u_3 be the other three vertices of the closed star of u_0 in X. If w_1 and w_2 are two other vertices of X, then (iii) implies that each 1-simplex meeting a w_i must also meet some u_j, $j > 0$. This accounts for three 1-simplices meeting each of these six vertices, so X contains no other 1-simplices or vertices. The 1-complex we have constructed is exactly $K_{3,3}$.

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