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EQUIVALENCE-SINGULARITY DICHOTOMIES FROM ZERO-ONE LAWS[1]

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ABSTRACT. In this note a general result on equivalence and singularity of two measures is presented. As a consequence of this S. Kakutani's dichotomy for product measures and J. Feldman's dichotomy for Gaussian measures are derived via appropriate zero-one laws.

In several different contexts, probability measures P, Q which are mutually absolutely continuous on each $\mathcal{F}_n$ of an increasing sequence of σ-algebras $\{\mathcal{F}_n, n = 1, 2, \cdots\}$ are then necessarily mutually absolutely continuous ($\equiv$) or singular ($\perp$) on the minimal σ-algebra containing $\bigcup_n \mathcal{F}_n$. Invariably, some type of 0-1 law is operative and suspected of forcing the equivalence-singularity dichotomy.

Here it is shown how in each case the dichotomy results from a certain tail σ-algebra being trivial. The natural technique for establishing triviality of this tail σ-algebra is none other than use of the appropriate 0-1 law.

THEOREM. *Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be σ-algebras of subsets of Ω, $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$ the σ-algebra generated by their union. For any probability measures P, Q on $\mathcal{F}$ which are mutually absolutely continuous when restricted to each of $\mathcal{F}_1, \mathcal{F}_2, \cdots$, we let $\rho_n = (dQ \mid \mathcal{F}_n)/(dP \mid \mathcal{F}_n)$ and define the tail algebra $\mathcal{G} = \bigcap_n \sigma\{\log \rho_{k+1} - \log \rho_k \mid k \geqq n\}$. The following is then true:*

$$(A \in \mathcal{G} \Rightarrow P(A) = 0 \text{ or } 1) \Rightarrow (P \perp Q \text{ or } P \ll Q).$$

PROOF. The measure Q has a Hahn decomposition by P which is

$$Q(A) = \int_A \rho \, dP + Q(AN) \quad \text{for every } A \in \mathcal{F}$$

where ρ is a nonnegative $\mathcal{F}$-measurable function and N is a P-negligible $\mathcal{F}$-measurable set. Using the submartingale convergence theorem it has

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been proved that $P(\lim_n \rho_n = \rho) = 1$ (e.g. [**1**, Lemma 3]). Observe that

$$Q(\rho_n = 0) = \int_{(\rho_n=0)} \rho_n \, dP = 0 \quad \text{for every } n$$

so that by mutual absolute continuity $P(\rho_n = 0) = 0$ for every n. Therefore $(\rho > 0)$ belongs to the P-completion of $\mathcal{G}$ since a.e. P,

$$\begin{aligned}(\rho > 0) &= \left(\lim_N \rho_N > 0\right) = \left(\lim_N \log \rho_{N+1} > -\infty\right) \\ &= \left(\lim_N \sum_{k=n}^{k=N} (\log \rho_{k+1} - \log \rho_k) > -\infty\right) \\ &\in \sigma\{\log \rho_{k+1} - \log \rho_k \mid k \geqq n\} \qquad \text{for every } n.\end{aligned}$$

If every event in $\mathcal{G}$ is P-trivial, then $P(\rho > 0) = 0$ or $P(\rho > 0) = 1$. In the former case

$$Q(N) = Q(\Omega) - \int_\Omega \rho \, dP = 1, \qquad P(N) = 0$$

so that $P \perp Q$. In the case $P(\rho > 0) = 1$, for every $A \in \mathcal{F}$ and $\alpha > 0$,

$$Q(A) \geqq \int_A \rho \, dP \geqq \int_{A(\rho > \alpha)} \rho \, dP \geqq \alpha P(A(\rho > \alpha))$$

and $P(A(\rho > \alpha)) \to P(A)$ as $\alpha \downarrow 0$. Hence $P(A) > 0$ implies $Q(A) > 0$. That is, $Q(A) = 0$ implies $P(A) = 0$. Therefore $P \ll Q$. □

COROLLARY. *With the same assumptions as in the Theorem,*

$$(A \in \mathcal{G} \Rightarrow P(A) = 0 \text{ or } 1 \text{ and } Q(A) = 0 \text{ or } 1) \Rightarrow (P \perp Q \text{ or } P \equiv Q).$$

PROOF. Interchanging the roles of P and Q does not alter $\mathcal{G}$ or the assumptions of the Theorem. □

Each of the two major types of equivalence-singularity dichotomies for measures will now be proved to follow via the Corollary. In every case, our method is the same: Use an appropriately chosen 0-1 law to establish triviality of $\mathcal{G}$ with respect to each of P and Q.

KAKUTANI'S THEOREM. *Suppose* $(\Omega, \mathcal{F}) = (\mathsf{X}_{k=1}^\infty \Omega_k, \otimes_{k=1}^\infty \mathcal{F}^{(k)})$ *and* $P = \mathsf{X}_{k=1}^\infty P^{(k)}$, $Q = \mathsf{X}_{k=1}^\infty Q^{(k)}$. *Then*

$$(P^{(k)} \equiv Q^{(k)} \text{ for every } k) \Rightarrow (P \perp Q \text{ or } P \equiv Q).$$

PROOF. For every n let $p^{(n)} = dQ^{(n)}/dP^{(n)}$,

$$\mathcal{F}_n = \bigotimes_{k=1}^{n} \mathcal{F}^{(k)} \otimes \bigotimes_{k=n+1}^{\infty} \{\phi, \Omega_k\},$$

then $\log \rho_{n+1} - \log \rho_n = \log \rho^{(n+1)}$. Hence $\mathcal{G}$ is the tail algebra of the

P-independent sequence $(\log \rho^{(n)}, n \geqq 1)$. By the ordinary 0-1 law [**5**, p. 229] we conclude $\mathcal{G}$ is P-trivial. Exactly the same argument applies for Q.

J. FELDMAN'S THEOREM. *Suppose T is a set (the time domain), $\mathcal{R}$ is the set of real numbers, $\Omega = \mathcal{R}^T$, $\mathcal{F} = \mathcal{B}^T$ (product Borel σ-algebra). Then*

$$(P, Q \text{ Gaussian measures on } \mathcal{F}) \Rightarrow (P \perp Q \text{ or } P \equiv Q).$$

PROOF. Let $X_t(f) = f(t)$, $t \in T$, be the coordinate random variables defined for $f \in R^T$. Suppose there is an $A \in \mathcal{B}^T$ with $0 = Q(A) < P(A)$. Then A is measurable [**6**, Corollary, p. 81] with respect to the σ-subalgebra generated by countable subfamily $\{X_{t_1}, X_{t_2}, \cdots\}$ of $\{X_t, t \in T\}$. Let $\mathcal{F}_\infty = \sigma\{X_{t_1}, X_{t_2}, \cdots\}$.

Define for each n, $\mathcal{F}_n = \sigma\{X_{t_1}, \cdots, X_{t_n}\}$. For each n, the dichotomy $P \mid \mathcal{F}_n \equiv Q \mid \mathcal{F}_n$ or $P \mid \mathcal{F}_n \perp Q \mid \mathcal{F}_n$ is a property of finite dimensional Gaussian distributions, so mutual absolute continuity $P \mid \mathcal{F}_n \equiv Q \mid \mathcal{F}_n$ may as well be assumed. For every $s, t \in T$ let

$$m_1(t) = \int_\Omega X(t)\, dP, \qquad K_1(s, t) = \int_\Omega X(x)X(t)\, dP - m_1(s)m_1(t),$$

$$m_2(t) = \int_\Omega X(t)\, dQ, \qquad K_2(s, t) = \int_\Omega X(s)X(t)\, dQ - m_2(s)m_2(t).$$

For each $n < \infty$ we define certain vectors and matrices by restriction to $T_n = \{t_1, \cdots, t_n\}$, e.g. $X_n = (X(t_1), \cdots, X(t_n))$, $K_{1,n} = [K_1(t_i, t_j); i, j \leqq n]$, etc. Mutual absolute continuity $P \mid \mathcal{F}_n \equiv Q \mid \mathcal{F}_n$ ensures for each $n < \infty$ that the rows of $K_{1,n}$ span the same linear submanifold of $\mathcal{R}^n$ as do the rows of $K_{2,n}$. For each $n < \infty$, $\rho_n = (dQ \mid \mathcal{F}_n)/(dP \mid \mathcal{F}_n)$ takes explicit form

$$\rho_n = \frac{|K_{1,\bar{n}}|^{1/2} \exp[-\frac{1}{2}(X - m_2)_{\bar{n}} K_{2,\bar{n}}^{-1} (X - m_2)_{\bar{n}}^{\mathrm{tr}}}{|K_{2,\bar{n}}|^{1/2} \exp[-\frac{1}{2}(X - m_1)_{\bar{n}} K_{1,\bar{n}}^{-1} (X - m_1)_{\bar{n}}^{\mathrm{tr}}]}$$

where $\bar{n}$ is the largest integer less than or equal to n with $|K_{1,\bar{n}}| > 0$. Rearrangement yields

$$\rho_n = \frac{|K_{1,\bar{n}}|^{1/2}}{|K_{2,\bar{n}}|^{1/2}} \exp[-\tfrac{1}{2}(X - m_2)_{\bar{n}}(K_{2,\bar{n}}^{-1} - K_{1,\bar{n}}^{-1})(X - m_2)_{\bar{n}}^{\mathrm{tr}}]$$
$$\times \exp[(X - m_1)_{\bar{n}} K_{1,\bar{n}}^{-1} (m_2 - m_1)_{\bar{n}}^{\mathrm{tr}} - \tfrac{1}{2}(m_2 - m_1)_{\bar{n}} K_{1,\bar{n}}^{-1} (m_2 - m_1)_{\bar{n}}^{\mathrm{tr}}].$$

In the reproducing kernel notation of [**7**],

$$\rho_n = |K_{2,\bar{n}}^{-1} K_{1,\bar{n}}|^{1/2} \exp \tfrac{1}{2}((X - m_2) \otimes (X - m_2), K_2 - K_1)_{K_{1,\bar{n}} \otimes K_{2,\bar{n}}}$$
$$\times \exp\{(X - m_1, m_2 - m_1)_{K_{1,\bar{n}}} - \tfrac{1}{2} \|m_2 - m_1\|^2_{K_{1,\bar{n}}}\}.$$

If $n < \infty$, define $\mathcal{M}_n$ as the set of all real functions on T which are finite real linear combinations of $\{K(\cdot, t_1), \cdots, K_1(\cdot, t_n)\}$, $\mathcal{M}_\infty = \bigcup_n \mathcal{M}_n$. Since $\mathcal{M}_\infty$ is dense in the reproducing kernel Hilbert space of the restriction $[K_1(t_i, t_j), i, j < \infty]$, P-triviality of $\mathcal{G}$ can be established using the following 0-1 law for Gaussian processes:

Zero-one law. If S is a set, $(\mathcal{R}^S, \mathcal{B}^S, P)$ a probability function space, P a Gaussian measure, A an event in the P-completion of $\mathcal{B}^S$, $\mathfrak{D}$ a dense subset of the reproducing kernel Hilbert space of the covariance kernel (e.g. K_1) of P, then

$$(e \in \mathfrak{D} \Rightarrow A = A + c) \Rightarrow (P(A) = 0 \text{ or } 1).$$

The proof will not be given here, but uses [**3**, Lemma 4] much as [**3**, Lemma 6]. This 0-1 law as applied to $\mathcal{G}$ asserts that $\mathcal{G}$ is P-trivial if invariant under the mapping $\{X(t_j) \to X(t_j) + e(t_j), j < \infty\}$ for every $e \in \mathcal{M}_\infty$. Such is indeed the case since if $e \in \mathcal{M}_n$, $k \geqq n$, simple calculation verifies that $\log \rho_{k+1} - \log \rho_k$ is *unchanged* when X_{k+1} is replaced by $(X + e)_{k+1}$ and X_k by $(X + e)_k$. Applying the Theorem, $P \mid \mathcal{F}_\infty \perp Q \mid \mathcal{F}_\infty$ or $P \mid \mathcal{F}_\infty \ll Q \mid \mathcal{F}_\infty$. Now $0 = Q(A) < P(A)$ precludes $\ll$, hence $P \mid \mathcal{F}_\infty \perp Q \mid \mathcal{F}_\infty$. Therefore (not $P \ll Q$) has been shown to imply $P \perp Q$. Interchanging the roles of P, Q we obtain $P \equiv Q$ or $P \perp Q$.

An effective test for singularity was provided by Kakutani in the product measure case, and later completely generalized by Kraft [**4**] who proved that

$$\left(\int_\Omega \rho_n^{1/2} \, dP \to 0 \right) \Leftrightarrow (P \perp Q).$$

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