PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 1, January 1972

# LOCALIZATION OF THE ZEROS OF THE PERMANENT OF A CHARACTERISTIC MATRIX[1]

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ABSTRACT. It is shown that many bounds for the eigenvalues of a complex matrix $A$ are also bounds for the zeros of the permanent of the characteristic matrix of $A$.

If $A = (a_{ij})$ is an $n$-square nonnegative matrix, let $\Phi(A)$ be the set of all $n$-square complex matrices $B = (b_{ij})$ for which $|b_{ij}| = a_{ij}$ whenever $i \neq j$, and let $\Psi(A)$ be the subset of $\Phi(A)$ consisting of all $B$ for which $|b_{ii}| = a_{ii}$ for all $i$. This note is based on the following.

THEOREM 1. *If $A$ is an $n$-square nonnegative matrix then*

$$\det B \neq 0 \quad \forall B \in \Psi(A) \quad \Rightarrow \quad \operatorname{per} B \neq 0 \quad \forall B \in \Psi(A). \tag{1}$$

PROOF. As Camion and Hoffman [4] have shown, if $\det B \neq 0$ for every $B \in \Psi(A)$, then there exist a permutation matrix $P$ and a nonsingular diagonal matrix $D$ such that $PAD$ is a dominant diagonal matrix. Brenner [1] has shown that the permanent of a dominant diagonal matrix is nonzero. Hence, since $PBD$ is dominant diagonal,

$$\operatorname{per} B = (\operatorname{per} PBD)(\operatorname{per} D)^{-1} \neq 0 \quad \forall B \in \Psi(A).$$

Consideration of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

shows that the converse of (1) is false.

If $A$ is an $n$-square matrix then each zero of $\operatorname{per}(zI - A)$ is called a *p-root* of $A$. Denote the set of all eigenvalues and the set of all $p$-roots of $A$ by $\Lambda(A)$ and $\Pi(A)$, respectively. Let $\delta(A)$ be the diagonal of the matrix $A$. An *admissible* mapping $\sigma$ of a set $S$ of $n$-square complex matrices is a

Received by the editors January 5, 1971.

*AMS* 1970 *subject classifications*. Primary 15A15, 15A42; Secondary 15A48.

*Key words and phrases*. Permanents, determinants, dominant diagonal matrices, bounds for eigenvalues, spectral radius, nonnegative matrices.

[1] This research was supported in part by the National Science Foundation and the Research Committee of the University of Alabama in Huntsville.

mapping of $S$ into the power set of the complex numbers such that if $A, B \in S$ with $\delta(A) = \delta(B)$ then $\sigma(A) = \sigma(B)$. We have the following.

THEOREM 2. *If $A$ is an $n$-square nonnegative matrix and $\sigma$ is an admissible mapping of $\Phi(A)$, then*

$$\Lambda(B) \subset \sigma(B) \quad \forall B \in \Phi(A) \quad \Rightarrow \quad \Pi(B) \subset \sigma(B) \quad \forall B \in \Phi(A).$$

PROOF. Let $B = (b_{ij}) \in \Phi(A)$, $z \notin \sigma(B)$, and let $A'$ be the $n$-square nonnegative matrix obtained from $A = (a_{ij})$ by replacing each $a_{ii}$ by $|z - b_{ii}|$. Let $C \in \Psi(A')$. Then, since $zI - B \in \Psi(A')$, there exists a unitary diagonal matrix $D$ such that $\delta(DC) = \delta(zI - B)$. Hence, there exists $B' \in \Phi(A)$ such that $\delta(B') = \delta(B)$ and $DC = zI - B'$. Since $z \notin \sigma(B) = \sigma(B')$,

$$\det C = (\det D)^{-1} \det DC = (\det D)^{-1} \det (zI - B') \neq 0.$$

Hence, $\det C \neq 0$ for every $C \in \Psi(A')$. Therefore, since $zI - B \in \Psi(A')$, by Theorem 1, $\operatorname{per}(zI - B) \neq 0$.

Many estimates for the eigenvalues of a matrix have been discovered (see for example, **[2]**, **[5]**, and their references). It follows from Theorem 2 that many of these are also estimates for the $p$-roots of the matrix. For example, we have the following.

COROLLARY 1. *If $A = (a_{ij})$ is an $n$-square complex matrix, $0 \leqq p \leqq 1$, and*

$$R_i = \sum_{k \neq i} |a_{ik}|, \quad C_i = \sum_{k \neq i} |a_{ki}|, \qquad i = 1, \cdots, n,$$

*then each $p$-root of $A$ lies in at least one of the ovals*

$$\{z : |z - a_{ii}|\, |z - a_{jj}| \leqq (R_i R_j)^p (C_i C_j)^{1-p}\}, \qquad 1 \leqq i < j \leqq n.$$

PROOF. Let $B = (b_{ij})$ be an $n$-square nonnegative matrix with $b_{ij} = |a_{ij}|$ whenever $i \neq j$. For each $D = (d_{ij}) \in \Phi(B)$, let

$$\sigma(D) = \bigcup_{1 \leqq i < j \leqq n} \{z : |z - d_{ii}|\, |z - d_{jj}| \leqq (R_i R_j)^p (C_i C_j)^{1-p}\}.$$

Using a theorem on eigenvalues due to Ostrowski **[6]**, we have

$$\Lambda(D) \subset \sigma(D) \quad \forall D \in \Phi(B).$$

Hence, since $A \in \Phi(B)$, from Theorem 2, $\Pi(A) \subset \sigma(A)$.

THEOREM 3. *Let $A$ be an $n$-square nonnegative matrix, and let $S$ be a set of complex numbers. Then*

$$\Lambda(B) \subset S \quad \forall B \in \Psi(A) \quad \Rightarrow \quad \Pi(B) \subset S \quad \forall B \in \Psi(A).$$

PROOF. Let $B \in \Psi(A)$. For each $C \in \Phi(A)$, let $\sigma(C)$ be $S$ or the set of all complex numbers according to whether $\delta(C) = \delta(B)$ or $\delta(C) \neq \delta(B)$. Now apply Theorem 2 to get $\Pi(B) \subset \sigma(B) = S$.

Let $\rho(A)$ be the spectral radius of the matrix $A$. Theorem 3 and a well-known bound for the eigenvalues of a matrix give the following.

COROLLARY 2. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be n-square matrices. If* $\pi$ *is a p-root of* $A$ *and* $|a_{ij}| \leqq b_{ij}$ *for* $i, j = 1, \cdots, n$, *then* $|\pi| \leqq \rho(B)$.

From this corollary we obtain the following result of Brenner and Brualdi [3].

COROLLARY 3. *If* $A$ *is an n-square nonnegative matrix and* $\pi$ *is a p-root of* $A$, *then* $|\pi| \leqq \rho(A)$.

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