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MORE ON THE SCHUR SUBGROUP

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ABSTRACT. Let k be an abelian extension of the rational field Q. We show Schur's subgroup $S(k)$ of the Bauer group $B(k)$ is usually of infinite index. Generators for p-torsion elements of $S(k)$ are found when k is the cyclotomic field of pth roots of unity.

Let k be an algebraic number field. We write $B(k)$ for the Brauer group of k, and $B_n(k)$ for the subgroup of $B(k)$ generated by classes of division rings of exponent n. Let $S(k)$ be the subgroup of $B(k)$ consisting of all classes which contain a simple component of $Q[G]$, the group algebra of a finite group G over the rational field Q. Following [6] we call $S(k)$ the Schur subgroup of k. Let $S_n(k) = S(k) \cap B_n(k)$. In [5] the structure of $S_3(k)$ for $k = Q(\surd(-3))$ is determined. Theorem 2 of this note generalizes the results of [5].

If A is a central simple algebra over k, we write $[A]$ for the corresponding class in $B(k)$.

THEOREM 1. *$S_n(k)$ is of infinite index in $B_n(k)$ for all $n \geqq 2$ unless $n = 2$ and $k = Q$. In the exceptional case $B_2(Q) = S_2(Q)$.*

PROOF. Since $S(k)$ is trivial unless k is the field of an irreducible character of a finite group G, we may assume k/Q is abelian. Assume $k \neq Q$ and $r = [k:Q]$. There are infinitely many primes of Q which split completely in k; let $p_1, \cdots, p_n, \cdots$ be an infinite list of them. For each p_i let $\mathfrak{g}_i^1, \cdots, \mathfrak{g}_i^r$ be the primes of k lying over p_i. We construct $[D_1], [D_2], \cdots, [D_m], \cdots$ in $B_n(k)$ as follows:

D_m is the central division ring over k whose Hasse invariants satisfy:

$$\operatorname*{inv}_{\mathfrak{g}^1_{2m-1}} D_m = \frac{1}{n}, \qquad \operatorname*{inv}_{\mathfrak{g}^1_{2m}} D_m = -\frac{1}{n},$$

$$\operatorname*{inv}_{\mathfrak{g}} D_m = 0 \quad \text{at all other primes } \mathfrak{g} \text{ of } k.$$

The construction of the D_m is allowed by [1, Theorem 7.8]. By [2] we

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have: $[D] \in S_n(k) \Rightarrow$ for each i, D has constant index at $\mathfrak{g}_i^1, \cdots, \mathfrak{g}_i^r$. Hence $[D_1], \cdots, [D_m], \cdots$ above represent distinct cosets of $S_n(k)$ in $B_n(k)$. It follows that $[B_n(k):S_n(k)] = \infty$.

If $k = Q$, then $S(k) = S_2(k)$ by the Brauer-Speiser theorem (see [**6**]). The fact that $B_2(Q) = S_2(Q)$ follows from [**4**].

Theorem 1 was also noted by Burton Fein.

Let p be a fixed odd prime. We will classify the algebras of index p in $S(k)$, where $k = Q(\xi_p)$ is the cyclotomic field of pth roots of unity. This generalizes Theorem 2 of [**5**].

If q is a prime, $q \equiv 1 \pmod p$, then q splits completely in $k = Q(\xi_p)$. Let $\mathfrak{g}_1, \cdots, \mathfrak{g}_{p-1}$ be the primes of k lying over q. The field $L = Q(\xi_q, \xi_p)$ is cyclic over $Q(\xi_p)$ of degree $q - 1$; let τ be the generator of the Galois group of L over k. Let H be the group generated by x, y, and z where $x^p = y^q = 1$, z acts on $\langle y \rangle$ according to the Galois action of τ on $Q(\xi_q)$, $z^{q-1} = x$, and x is central in H. Then the cyclic algebra $\mathfrak{A} = (k(\xi_q), \tau, \xi_p)$ is a homomorphic image of $Q[H]$, so $[\mathfrak{A}]$ is in the Schur subgroup of k. Clearly $\mathfrak{A}^p$ is a total matrix algebra since $\xi_p^p = 1$; so $[\mathfrak{A}]$ has order 1 or p in $B(k)$. $[\mathfrak{A}]$ has order $1 \Leftrightarrow \xi_p$ is a norm from $L = k(\xi_q)$ to k. We show ξ_p is not a local norm at the primes $\mathfrak{g}_1, \cdots, \mathfrak{g}_{p-1}$ above. For convenience we fix $\mathfrak{g} = \mathfrak{g}_1$.

The extension L/k is totally and tamely ramified at $\mathfrak{g}$; let t be the unique prime of L lying over $\mathfrak{g}$. If U_t (resp. $U_\mathfrak{g}$) denotes the units of L_t (resp. $k_\mathfrak{g}$) and U_t^1 (resp. $U_\mathfrak{g}^1$) those which are 1 (mod t) (resp. 1 (mod $\mathfrak{g}$)), then as in [7, v, #3] the norm induces a homomorphism:

$$N_0: U_t/U_t^1 \to U_\mathfrak{g}/U_\mathfrak{g}^1. \tag{1}$$

But $U_t/U_t^1 \simeq \bar{L}_t^*$, the multiplicative group of the residue class field of L at t. Similarly $U_\mathfrak{g}/U_\mathfrak{g}^1 \simeq \bar{k}_\mathfrak{g}^* \simeq \bar{L}_t^* \simeq Z_q^*$ as $\mathfrak{g}$ is totally ramified in L. Thus (1) reduces to a homomorphism:

$$N_0: Z_q^* \to Z_q^* \tag{2}$$

of cyclic groups of order $q - 1$. By [7, Proposition 5, p. 92] we have: $N_0(x) = x^{q-1}$ in (2). Hence the image of N_0 is trivial, so N_0 does not cover the image of ξ_p; it follows that ξ_p is not a norm.

Thus $[\mathfrak{A}]$ represents an element of order p in $S(k)$. Clearly $\mathfrak{A}$ is split at all primes w, $w \notin \{\mathfrak{g}_1, \cdots, \mathfrak{g}_{p-1}\}$, for each such prime is unramified from k to L and so ξ_p is a unit, hence a norm, at w. By the proof of Theorem 5 of [**3**] we have with suitable relabelling, invariants of $[\mathfrak{A}]$ of form $1/p$, $2/p, \cdots, (p-1)/p$ at $\mathfrak{g}_1, \cdots, \mathfrak{g}_{p-1}$.

We claim $S_p(k)$ is generated by the classes $[\mathfrak{A}]$ above. Suppose D is a central division algebra over k with $[D] \in S_p(k)$; D has exponent p. b If is a rational prime, $q \equiv 1 \pmod p$, and $\mathfrak{g}_1, \cdots, \mathfrak{g}_{p-1}$ are the primes of

$k = Q(\xi_p)$ lying over q, we have by [2], $\text{inv}_{\mathfrak{g}_1} D = 0 \Rightarrow \text{inv}_{\mathfrak{g}_i} D = 0$, $i = 1, \cdots, p-1$. Assume $\text{inv}_{\mathfrak{g}_1} D = a/p$, $(a, p) = 1$. Set $x = [\mathfrak{A}]^{-a} \cdot [D] \in S_p(k)$; then x has invariant 0 at $\mathfrak{g}_1 \Rightarrow \text{inv}_{\mathfrak{g}_i} x = 0$ for $i = 1, \cdots, p-1 \Rightarrow \text{inv}_{\mathfrak{g}_i} D = ai/p = i(a/p)$. Thus D has invariants of type $1/p$, $2/p, \cdots$, $(p-1)/p$ at split primes over q where $q \equiv 1 \pmod p$. We must show that D has no other non-0 invariants. By appropriate multiplication in $B(k)$ as above we may assume $\text{inv}_{\mathfrak{g}} D = 0$ for all q lying over completely split primes of Q.

D has no non-0 invariants at primes of k lying over odd rational primes by [8, Satz 10]. Also, since the index of D is p and $p \neq 2$, then D has no non-0 invariants at primes of k extending 2 by [8, Satz 11]. We have proved:

THEOREM 2. *If p is an odd prime, then the division rings D with $[D] \in S_p(k)$, $k = Q(\xi_p)$ have invariants of type $1/p$, $2/p, \cdots, (p-1)/p$ at completely split primes of k, and 0 everywhere else. The classes $[\mathfrak{A}]$ above generate $S_p(k)$.*

We note that Theorem 2 has the following unusual consequence: If p and q are distinct odd primes, then ξ_p is not a norm from $Q(\xi_p, \xi_q)$ to $Q(\xi_p) \Leftrightarrow q \equiv 1 \pmod p$.

Many thanks are due to both Burton Fein and Basil Gordon for pointing out to me the existence and applicability of [8]. Ken Fields has noted that there is no way of determining $S_{p^2}(Q(\xi_{p^2}))$ without first determining $S_p(Q(\xi_{p^2}))$.

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