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ON THE SMOOTHNESS OF EIGENFUNCTIONS OF HYPONORMAL SINGULAR INTEGRAL OPERATORS

KEVIN CLANCEY[1]

ABSTRACT. Fix $\varphi \in L^\infty(E)$ and let $E \subset \boldsymbol{R}$ be bounded and measurable; for $1<p<\infty$ consider the bounded linear operator

$$Tf(s) = sf(s) + \frac{\varphi(s)}{\pi}\int_E^* \frac{\bar{\varphi}(t)f(t)}{t-s}\,dt \quad \text{a.e.} \quad s \in E$$

where $f \in L^p(E)$. If $\nu=\lambda+i\mu \in \boldsymbol{C}$ then there are no nonzero $L^p(E)$ solutions of $Tf=\nu f$ for $p>2$ in case λ is a point of positive Lebesgue density in the complement of E.

1. **Introduction.** For φ fixed in $L^\infty(E)$ where $E \subset \boldsymbol{R}$ is bounded and measurable, the operator

$$(1) \quad Tf(s) \equiv T_\varphi f(s) = sf(s) + i\left(\frac{\varphi(s)}{\pi i}\int_E^* \frac{\bar{\varphi}(t)f(t)}{t-s}\,dt\right) \text{a.e. } s \in E, f \in L^p(E),$$

is a bounded operator on $L^p(E)$ for $1<p<\infty$. The singular integral is to be considered as a Cauchy principal value, that is, $\int^* = \lim_{\varepsilon \downarrow 0} \int_{|s-t| \geqq \varepsilon}$. In the case $p=2$ the operator T^* (the adjoint of T) belongs to the class of hyponormal operators. An operator A on a Hilbert space is called hyponormal in case its selfadjoint self-commutator $A^*A-AA^*=D$ is positive semidefinite ($D \geqq 0$). These hyponormal singular integral operators were studied by Putnam [**1**] and Clancey [**2**]. When φ is Riemann integrable the spectrum of T_φ was computed by Putnam in [**1**]. In the case where φ is only assumed to be L^∞ the spectrum was described in [**2**]. The knowledge of the fine structure of the spectrum is limited. In case φ is smooth (say continuous) and $p=2$ it is possible to describe which parts of the spectrum correspond to point spectrum of the operator T_φ (see Putnam [**1**] and also Tricomi [**3**, p. 190]). In Tricomi, one gets an idea of how smooth the eigenfunctions are. Putnam [**1**] also proves in the case $p=2$ that if $|\varphi(x)| \geqq \gamma > 0$ a.e. on an interval $[\alpha, \beta] \subset E$, then $\xi \in (\alpha, \beta)$ belongs to the

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point spectrum of T_φ. One reason for interest in the fine structure of the spectrum of T is the fact that existence of point spectrum of an operator or its adjoint automatically gives an invariant subspace theorem for the operator. In spite of the fact that the self-commutator of an operator T of the form (1) is one dimensional there is no theorem stating that these operators have an invariant subspace.

Some positive results can be stated. For example, if $|\varphi(x)| \geqq \gamma > 0$ a.e. on $[\alpha, \beta] \subset E$ then by Putnam's result T has an invariant subspace. Our results will show that eigenfunctions of the equation $Tf = \nu f$ (if they exist) will be nonsmooth. The proof of our result will indicate an unpleasant (perhaps unnecessary) association of the problem of the fine structure of the spectrum of the operators in (1) with harmonic conjugate function theory for L^1 functions.

2. The operator T acting on $L^2(E)$ can be written in the cartesian form $T = H - iJ$ where H and J are the selfadjoint operators defined for $f \in L^2(E)$ by $Hf(s) = sf(s)$ and $Jf(s) = (\varphi(s)/\pi i) \int_E^* \bar{\varphi}(t) f(t)(s-t)^{-1}\, dt$ a.e. $s \in E$. For an operator A acting on some complex linear space X we will write $A_\nu = A - \nu I$ where I is the identity on X and $\nu \in \boldsymbol{C}$. The spectrum of an operator T on a Hilbert space will be denoted by $\sigma(T)$. For $E \subset \boldsymbol{R}$ measurable define the essential closure of E as the set $\bar{E}^{\text{ess}} = \{x \in \boldsymbol{R} \mid \text{meas}_1((x-\delta, x+\delta) \cap E) > 0 \text{ for all } \delta > 0\}$, where meas_1 denotes Lebesgue measure on $\boldsymbol{R}$. A set $E \subset \boldsymbol{R}$ will be called perfect in measure in case $\bar{E}^{\text{ess}} = E$. The spectrum of the operators T of the form (1) on $L^2(E)$ was described in [**2**] in

THEOREM 1. *If T is the operator defined by* (1) *on $L^2(E)$, then* $\sigma(T) = \{x + iy : x \in \bar{E}^{\text{ess}}, -M_x \leqq y \leqq M_x\}$ *where* $M_x = \text{ess lim sup}_{t=x}[|\varphi(t)|^2]$.

A hyponormal operator A on a Hilbert space $\mathfrak{H}$ is called completely hyponormal in case there are no subspaces reducing A (that is invariant under A and A^*) on which A is a normal operator. In case $\varphi(t) \neq 0$ a.e. $t \in E$, then the adjoint of T defined by (1) is completely hyponormal on $L^2(E)$. (See, e.g. Putnam [**1**].) We will assume φ has this property. Since A hyponormal on $\mathfrak{H}$ is equivalent to $\|A_\nu f\|^2 \geqq \|(A_\nu)^* f\|^2$ for all $f \in \mathfrak{H}$ and $\nu \in \boldsymbol{C}$ it follows that eigensubspaces of a hyponormal operator reduce the operator; moreover, the restrictions of the operator to these subspaces are normal operators. Then there are no nonzero $L^2(E)$ solutions of $(T_\nu)^* f = 0$ where T is given by (1) and $\nu \in \boldsymbol{C}$; hence there are no $L^{2+\delta}(E)$ solutions of $(T_\nu)^* f = 0$, for any $\delta > 0$.

The following lemma is a simple corollary of M. Riesz's theorem which states that the Hilbert transform $Qf(s) = (1/\pi i) \int_{\boldsymbol{R}}^* f(t)(t-s)^{-1}\, dt$ is a bounded linear operator on $L^p(\boldsymbol{R})$ for $1 < p < \infty$.

LEMMA 1. *If* $f \in L^p(\boldsymbol{R})$ *where* $1<p<\infty$ *then*

$$\text{meas}_1[\{s \in \boldsymbol{R} : |Qf(s)| \geqq \sigma > 0\}] \leqq A_p \, \|f\|_p^p/\sigma^p$$

where A_p *depends only on* p.

PROOF. See for example Katznelson [**4**, Chapter 3].

We present the main result.

THEOREM 2. *Let* E *be bounded and measurable. Then for* $\delta>0$ *there are no nonzero* $L^{2+\delta}(E)$ *solutions of* $T_\nu f=0$ *where* T *is the operator defined by* (1) *in case* $\lambda=\text{Re}\,\nu$ *is a point of positive Lebesgue density in the complement of* E.

PROOF. We will give a proof first for the case where E is perfect in measure. If $x \notin E=\bar{E}^{\text{ess}}=\sigma(H)$, then since $0\leqq(Cf, f)=(i[H_xJ_y-J_yH_x]f,f)$ for $f\in L^2(E)$ and $x, y\in \boldsymbol{R}$ it follows that

$$(2) \qquad 0 \leqq (CH_x^{-1}f, H_x^{-1}f) = (i[J_yH_x^{-1} - H_x^{-1}J_y]f, f), \qquad f \in L^2(E).$$

Now when $H_\lambda f=iJ_\mu f$ for $f\neq 0$ in $L^2(E)$, one obtains, from (2), $2(H_\lambda H_x^{-1}f,f)\leqq 0$ or $2\int_E (t-\lambda)(t-x)^{-1}|f(t)|^2\,dt\leqq 0$, $x\notin\sigma(H)$. This implies

$$(3) \qquad \|f\|_2^2/\pi\,|x-\lambda| \leqq |Q(|f|^2)(x)|, \qquad x \notin E.$$

Now by Lemma 1 if the nonzero solution of $H_\lambda f=iJ_\mu f$ is in $L^{2+\delta_0}(E)$ for $\delta_0>0$, then

$$\text{meas}_1[(R\backslash E) \cap [\lambda-\varepsilon, \lambda+\varepsilon]] \leqq A_{P_0}\pi^{P_0}\varepsilon^{P_0}\frac{\|\,|f|^2\|_{P_0}^{P_0}}{\|f\|_2^{2P_0}},$$

where $P_0 = 1 + \delta_0/2$

One concludes that

$$\lim_{\varepsilon\downarrow 0}\frac{\text{meas}_1[(R-E)\cap[\lambda-\varepsilon,\lambda+\varepsilon]]}{2\varepsilon} = 0.$$

A trivial modification of the preceding argument can be made to conclude

$$\lim_{h\downarrow 0, k\downarrow 0}\frac{\text{meas}_1[(R-E)\cap[\lambda-h,\lambda+k]]}{h+k} = 0.$$

This completes the proof in the case where E is perfect in measure.

We now derive an analogue of equation (2) when x is in the spectrum of H. However, the integral in (2) must be replaced by a Cauchy principal value. Assume now $H_\lambda f = iJ_\mu f$ for $f \neq 0$ in $L^{2+\delta}(E)$ where $\delta > 0$. Then

$$
\begin{aligned}
\frac{2}{\pi}\int_E^* \frac{t-\lambda}{t-x}|f(t)|^2\,dt &= \frac{1}{\pi}\int_E^* \frac{f\overline{iJ_\mu f}}{t-x}\,dt + \frac{1}{\pi}\int_E^* \frac{iJ_\mu f\bar f}{t-x}\,dt \\
&= -\left[\frac{1}{\pi^2}\int^* \frac{\varphi\bar f(t)}{t-x}\,dt\int^* \frac{\bar\varphi f(s)}{s-t}\,ds + \frac{1}{\pi^2}\int^* \frac{\bar\varphi f(t)}{t-x}\,dt\int^* \frac{\varphi\bar f(s)}{s-t}\,ds\right] \qquad (4)\\
&= -\left|\frac{1}{\pi}\int^* \frac{\varphi\bar f}{t-x}\,dt\right|^2 + |\varphi f(x)|^2 \quad \text{a.e.}
\end{aligned}
$$

The last equality follows since φf is in $L^{2+\delta}(E)$ by Theorem IV of Tricomi [3, p. 169]. From (4) one obtains

$$\|f\|^2 \leqq |\lambda - x|\left|\int^* \frac{|f|^2}{t-x}\,dt\right| \quad \text{a.e.} \quad x \notin E$$

and the proof of Theorem 2 when E is not perfect in measure follows as above.

3. The following result due to Putnam [5] (see also Stampfli [6] and Sz.-Nagy and Foias [7, p. 93]) can be used to rule out nonzero $L^2(E)$ (hence $L^{2+\delta}(E)$) solutions of $T_\nu f = 0$. For K compact in $\boldsymbol{C}$ (respectively $\boldsymbol{R}$) define

$$
\begin{aligned}
\gamma_K(x_0) &= \sup_{x\notin K} \frac{d(x, K)}{d(x, x_0)}, & x_0 \in K,\\
&= 1, & x_0 \notin K,
\end{aligned}
$$

where d denotes distance in $\boldsymbol{C}$ (respectively $\boldsymbol{R}$).

THEOREM 3. *Let A be completely hyponormal on $\mathfrak{H}$, $\sigma = \sigma(T)$ then if $\gamma_\sigma(\nu) = 1$ there is no nonzero solution of $(A_\nu)^* f = 0$.*

Consequently, if E is perfect in measure there are no nonzero $L^2(E)$ solutions of $T_\nu f = 0$ when $\gamma_E(\operatorname{Re}\nu) = 1$. It should be remarked that $\gamma_E(\operatorname{Re}\nu) = 1$ implies the Lebesgue density of $\operatorname{Re}\nu$ in $\boldsymbol{R}\backslash E$ is positive.

The author would like to thank Alexander Davie for a conversation concerning the following lemma.

LEMMA 2. *Let $E \subset \boldsymbol{R}$ be compact, then* $\operatorname{meas}_1[\{x \mid \gamma_E(x) = 1\} \cap E] = 0$.

PROOF. One can write $\boldsymbol{R}\backslash E = \bigcup_{n=1}^\infty (a_n, b_n)$ where (a_n, b_n) are pairwise disjoint. Set $c_n = (a_n + b_n/2)$, $r_n = (b_n - a_n)/2$; for $\eta > 1$, let $E_\eta = \boldsymbol{R}\backslash\bigcup_{n=1}^\infty (c_n - \eta r_n, c_n + \eta r_n)$. Then $\lim_{\downarrow 1} \operatorname{meas}_1 E_\eta = \operatorname{meas}_1 E$ and $x_0 \in E_\eta$ implies, for $x \in (a_n, b_n)$,

$$d(x, E)/(x, x_0) \leqq d(c_n, E)/d(c_n, x_0) \leqq d(c_n, E)/d(c_n, E_\eta) = 1/\eta.$$

Consequently $\gamma_E(x_0) \leqq 1/\eta$ for $x_0 \in E_\eta$ and the lemma follows.

Since the spectrum of T_φ has positive measure, it follows from Lemma 2 that Theorem 3 does not preclude the possibility of point spectrum of the operators T_b. The possibility of nonzero $L^2(E)$ solutions of $T_\nu f=0$ for T defined by equation (1) and ν in the boundary of $\sigma(T)$ remains unanswered.

One interesting example is the case where E is a Cantor set K of positive measure and $\varphi=1_K$. Theorem 2 shows that as an operator on $L^{2+\delta}(K)$ for $\delta>0$, T_φ has no eigenvalues. A similar result is true if $E=G$ and $\varphi=1_G$ where G is a union of perfect nowhere dense sets in $[-1, 1]$ with $\text{meas}_1 G=1$ and the Lebesgue density of every point in $[-1, 1]$ in the complement of G is positive.

In the case where $E=[a, b]$ and φ is smooth, Tricomi [**3,** p. 185] obtains nonzero L^1 solutions of $T_a f=0$.

ADDED IN PROOF. The author has recently obtained examples of operators of the form (1) having $L^2(E)$ eigenvalues in the boundary of the spectrum.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024

Current Address: Department of Mathematics, University of Georgia, Athens, Georgia 30601