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SETS ACCESSIBLE AT EACH POINT ONLY BY WILD ARCS

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ABSTRACT. A positional pathology as described in the title is shown to occur in three-space. We construct an arcwise accessible point set M such that each arc to M is locally knotted at uncountably many points. In addition we give examples of connected, locally connected point sets S and T which are accessible at each point only by wild arcs and tame arcs respectively. The point set M has countably many components, and each of these is a tame finite 2-complex. Moreover E^3-M is locally connected and arcwise connected.

1. **Introduction.** An important segment of the work dealing with the position of a subset in a space is concerned with accessibility. Early references and historical comments, beginning with Schoenflies' 1908 paper, may be found in [**11**]. A point x of a space X is said to be *accessible from a subset* Y *of* X if for each y in Y there is an arc A from x to y such that $A-x\subset Y$. We say y is an *accessible point* of a subset Y of X if y is accessible from $X-Y$. In more recent work, stronger accessibility properties such as accessibility by tame arcs and piercing by tame arcs characterize tame embeddings of certain sets in E^3 ([**3**], [**9**]).

We say a closed set X in E^3 is *tame* if there is a homeomorphism of E^3 onto itself which carries X onto a polyhedron (the union of a locally finite collection of tetrahedra, triangles, segments and points). X is *wild* if it is not tame. The simplest set which can be tame or wild is an arc. An arc A is *locally unknotted at* a provided some neighborhood of a in A lies in a topological disk. A is *locally unknotted* if it is locally unknotted at each point. Recently Keldyš has shown that an arc is locally unknotted iff it lies in a topological disk [**8**]. Local unknottedness is one of two independent properties which characterize tame arcs [**2**], [**7**].

Here we investigate the positional property of a subset being accessible only by a restricted class of arcs. The existence of a set accessible only by wild arcs follows from Bing's example of a simple closed curve which pierces no disk [**1**].

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EXAMPLE 1. *A connected locally connected accessible subset S of E^3 each of whose points is accessible only by wild arcs.* Let $r_1, r_2, \cdots$ be the rational points in E^3. Let A_{ij} be an arc from r_i to r_j such that A_{ij} pierces no disk and diameter A_{ij} is the distance from r_i to r_j. Then $S=E^3-UA_{ij}$ is the desired set.

In §2, we construct an accessible point set M as described in the abstract. The construction involves inductively positioning disjoint 2-complexes around auxiliary knotted arcs to form a labyrinth. Each arc to a point of the set must traverse its knotted passages in such a manner that the arc contains sequences of knotted subarcs converging to uncountably many points of the arc. In §3, we show M has the desired properties. Finally in §4 we discuss related positional questions and give an example of a set accessible only by tame arcs.

Point sets which have the positional properties discussed here cannot be locally compact at any point. If X is a locally compact subset of E^3 with empty interior, then X contains a dense set of points accessible by tame arcs and a dense set of points accessible by wild arcs. The proof of this is immediate.

2. **A labyrinth.** The desired set $M=\bigcup_{n=1}^{\infty} M_n$ where the sets M_n are defined inductively as follows.

Let C be the cylinder $D \times I$ where D is the interior of the unit circle in the plane and I is the closed unit interval. Let B be a triangle together with its interior. A *prismatic cell* is a homeomorphic image of $P_0=B\times I$. For each prismatic cell P, choose a homeomorphism F_P from P_0 onto P. The *triangular* and *rectangular faces* of P are then the images of the respective faces of P_0 under F_P.

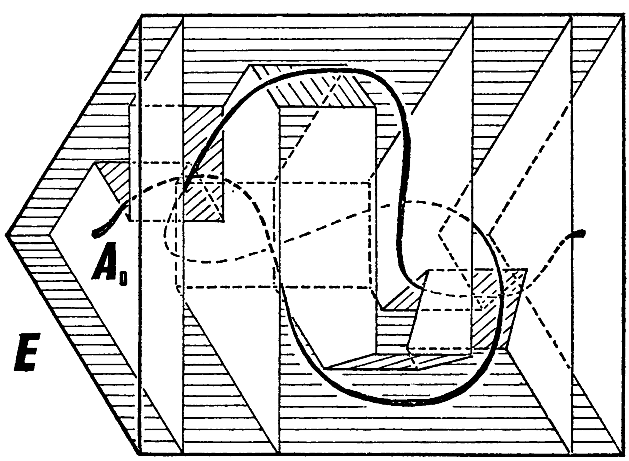

FIGURE 1

Let A_0 be the knotted arc depicted in Figure 1. A_0 passes through the interior of P_0 and has end points which are interior points of the triangular faces.

Let E be the 2-complex depicted in Figure 1. E consists of the three rectangular faces of P_0 together with the obstructions depicted in the interior of P_0. The interiors of the two triangular faces of P_0 are not in E.

There is then a homeomorphism G from C onto P_0-E with the following properties. G carries the axis of C onto A_0. Each arc which lies wholly in C except for an end point which is in the cylinder bounding C is carried by G onto an arc lying wholly in P_0-E except for an end point in E. Finally each point of E is the end point of such an arc. If P is any prismatic cell, let F_P^* denote the homeomorphism $F_P \cdot G$ from C into P.

In the following, no special properties of the knotted arc A_0 are used. Instead any knotted arc can be used as long as E and G can be chosen as above.

Let Q denote the set of all points of the Cantor ternary set which are not end points of removed middle third intervals. Let $\mathscr{S}$ denote the collection of all components of the open unit ball minus all points x such that the distance from x to the origin is a number in Q. Then $\mathscr{S}$ is a countable collection of closed concentric spherical shells. Let $\mathscr{T}$ denote the collection of all components of the cylinder C minus all points x such that the distance from x to the axis of C is a number in Q. Then $\mathscr{T}$ is a countable collection of closed tubular shells.

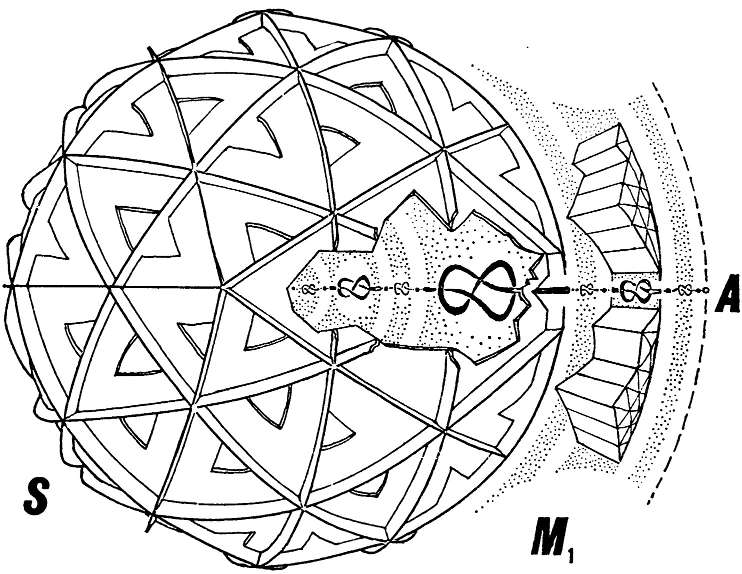

FIGURE 2

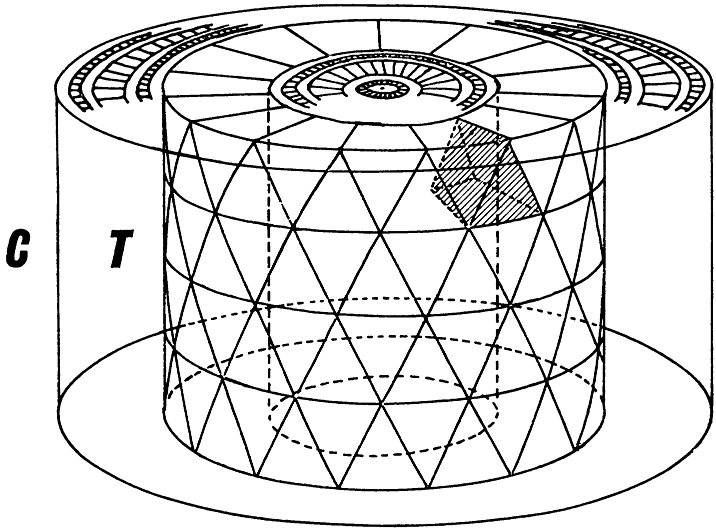

FIGURE 3

Initial stage. Form a finite 3-complex in each spherical shell in $\mathscr{S}$ as follows. Let S be a member of $\mathscr{S}$. Triangulate the outer surface of S with triangles having diameter less than the thickness of S and such that the edge of each triangle lies in a plane through the origin. Form prismatic cells in S using these planes.

Let $\mathscr{P}_1$ denote the collection of all prismatic cells formed in this way in each shell in $\mathscr{S}$. Let $M_1=\bigcup \{F_P(E):P\in\mathscr{P}_1\}$. Each component of M_1 is then a finite 2-complex in a shell S in $\mathscr{S}$ as depicted in Figure 2.

Transition. Assume the construction through the nth stage has been performed.

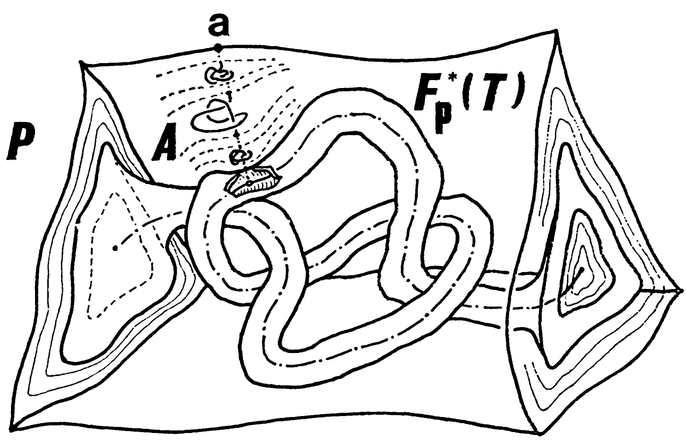

FIGURE 4

Form a finite 3-complex on each tubular shell T in $\mathscr{T}$ as follows. Triangulate the outer surface of T with triangles having diameter less than the thickness of T. Then form prismatic cells using lines perpendicular to the axis of C. This is depicted in Figure 3.

For each prismatic cell P in $\mathscr{P}_n$, let $\mathscr{P}_{n+1}(P)$ be the collection of all prismatic cells $F_P^*(X)$ where X is a prismatic cell as above in some tubular shell in $\mathscr{T}$. Then $\bigcup \mathscr{P}_{n+1}(P)$ is a point set in P whose components are finite 3-complexes. Finally let $\mathscr{P}_{n+1}$ be the collection of all members of $\mathscr{P}_{n+1}(P)$ for P in $\mathscr{P}_n$, and let $M_{n+1}=\bigcup \{F_P(E):P\in\mathscr{P}_{n+1}\}$.

M_{n+1} is then a countable union of homeomorphic copies of E such that if two intersect, their intersection is a rectangular face of each. M_{n+1} does not intersect M_i for $i<n$, and M_{n+1} does not intersect the auxiliary knotted arcs $F_P(A_0)$ for P in $\mathscr{P}_i$ where $i\leq n+1$. Moreover each component of M_{n+1} is a finite 2-complex in some P in $\mathscr{P}_n$. Such a component lies in a knotted tubular shell $F_P^*(T)$, as depicted in Figure 4, where T is in $\mathscr{T}$.

This completes the inductive construction of M.

Select the triangulations and maps in the previous construction so that the copies of P_0 and E are tame. The components of M are then tame finite 2-complexes.

3. **Properties of the labyrinth.**

PROPOSITION 3.1. *Each point of M is accessible and E^3-M is arcwise connected.*

PROOF. Let $\mathscr{K}_1$ be the collection of all knotted arcs $F_P(A_0)$ for P in $\mathscr{P}_1$, and let $H_1=\bigcup \mathscr{K}_1\cup(E^3-\bigcup \mathscr{S})$. Clearly each member of $\mathscr{K}_1$ is a subarc of an arc A as depicted in Figure 2 which lies in H_1 and joins the origin to a point of the unit sphere. A intersects each component of $E^3-\bigcup \mathscr{S}$, and each of these is a sphere with radius in Q. H_1 is then arcwise connected.

Let n be a positive integer, and let P be a member of $\mathscr{P}_n$. Let $\mathscr{K}(P)$ be the collection of all knotted arcs $F_X(A_0)$ for $X\in\mathscr{P}_{n+1}(P)$. Let $H(P)=F_P^*(C-\bigcup \mathscr{T})$. $H(P)$ then consists of the knotted arc $F_P(A_0)$ together with concentric tubular surfaces each the image of a cylinder in C which has radius a nonend point of the Cantor set.

Clearly each member of $\mathscr{K}(P)$ is a subarc of an arc A as depicted in Figure 4. A lies in $W(P)=H(P)\cup\bigcup \mathscr{K}(P)$ except for one end point a and joins a point of the central knotted arc $F_P(A_0)$ to a point a of the 2-complex $F_P(E)$. A then intersects each of the components of $H(P)$. Since each of these components is arcwise connected and each member of $\mathscr{K}(P)$ lies in an arc like A, $W(P)$ is arcwise connected. Moreover each point of

$F_P(E)$ is the end point of an arc like A. Thus each point of $F_P(E)$ is accessible from $W(P)$.

Let $H_{n+1}=\bigcup \{W(P):P\in\mathscr{P}_m \text{ and } m\leqq n\}$ for n a positive integer. Since H_1 is arcwise connected and $W(P)$ intersects H_1 for each P in $\mathscr{P}_1$, H_2 is arcwise connected and each point of M_1 is accessible from H_2. Assume H_n is arcwise connected. For each P in $\mathscr{P}_n$, $W(P)$ contains $F_P(A_0)$ which is also in H_n. Therefore H_{n+1} is arcwise connected, and each point of M_n is accessible from H_{n+1}.

Let $H=\bigcup_{n=1}^{\infty} H_n$. Then H is arcwise connected, and each point of M is accessible from H.

Suppose there is an x in the complement of M which is not in H. Then there is a monotone collection $\{P_n\}$ of prismatic cells such that $P_n\in\mathscr{P}_n$ and $x=\bigcap \{P_n\}$. Let x_n be a point of $F_{P_n}(A_0)$ for each n. For each n there exists an arc A_n from x_n to x_{n+1} in $H_{n+1}\cap P_n$ and which intersects $F_{P_{n+1}}(A_0)$ in x_{n+1}, e.g., a subarc of the central arc $F_{P_n}(A_0)$ in P_n together with a subarc of an arc A as depicted in Figure 4. Then $A^*=x+\bigcup_{n=1}^{\infty} A_n$ is an arc, and A^* lies in H except for x. Thus E^3-M is arcwise connected. Since H lies in E^3-M and M is accessible from H, M is accessible from E^3-M. This completes the proof.

Let A be an arc and P be a prismatic cell. Let n be the number of components K of $A\cap P$ such that K is a subarc of A and the end points of K are on opposite triangular faces of P. We say *A spans the knotted passage of P* if n is odd and $A\cap F_p(E)=\varnothing$.

PROPOSITION 3.2. *Each arc to M is locally knotted at uncountably many points.*

PROOF. Let $x\in M$ and let A be an arc lying wholly in E^3-M except for its end point x. There is an n and an X in $\mathscr{P}_n$ such that $x\in F_X(E)$. Suppose x is not a point of one of the two triangular faces of X. Then there is a nonend point $z\in A$ such that zx lies wholly in P where P is X or a member of $\mathscr{P}_n$ contiguous to X. There is an $r_0<1$ such that the distance from $F_P^{*-1}(z)$ to the y-axis is less than r_0. Let $Q_0=Q\cap(r_0, 1)$. For each $r\in Q_0$, zx intersects $T(r)$, the image of the cylindrical surface about the y-axis with radius r under F_P^*. Let $r\in Q_0$ and let $r_1>r_2>\cdots$ be a sequence of members of Q_0 converging to r. Let K_i be a component of $\bigcup \mathscr{P}_{n+1}(P)$ between $T(r_i)$ and $T(r_{i+1})$. Each K_i separates x and z in P. Therefore there is a $P_i\in\mathscr{P}_{n+1}(P)$ such that $P_i\subset K_i$ and zx spans the knotted passage of P_i for each i. There is a subsequence $P_1^*, P_2^*, \cdots$ of $P_1, P_2, \cdots$ which converges to a point $y(r)\in zx\cap T(r)$. Let $Y=\{y(r):r\in Q_0\}$. Then Y is uncountable. If x is a point of a triangular face of P, then, by a similar argument, there is a point $z\in A$ and an uncountable subset Y of interior points of zx such that for each $y \in Y$ there is some sequence $P_1^*, P_2^*, \cdots$ of prismatic cells converging to y such that zx spans the knotted passage of each P_i^*.

Now suppose A is locally unknotted at $y \in Y$. Then there is a topological disk D_1 and a subarc A_1 of A such that $A_1 \subset D_1$ and $y \in \operatorname{Int} A_1$. There is an n and a subarc A_2 of $zx \cap A_1$ such that $y \in \operatorname{Int} A_2$ and A_2 spans the knotted passage of P_i^* for all $i > n$. A_2 lies in the boundary of a subdisk D_2 of D_1. There is an m such that $\partial D_2 \cap P_m^* = A_2 \cap P_m^*$. Let $W = P_m^*$. There is a polyhedral disk D and a subarc A^* of ∂D such that $\partial D \cap W = A^* \cap W$ and A^* spans the knotted passage of W.

Let $r_0 < 1$ be such that the solid cylinder $C_0 = \{u \in C$: distance u to y-axis $\leqq r_0\}$ contains $F_W^{*-1}(\partial D \cap W)$ in its interior relative to C. Let U_1 and U_2 denote the interiors of the two end disks of C_0. Let T be the torus $F_W^*(\partial C_0 - U_1 \cup U_2) \cup \partial W - F_W^*(U_1 \cup U_2)$. Then T bounds a solid torus V_1 in S^3, the one point compactification of E^3, such that $\partial D \subset \operatorname{Int} V_1$ and the core of V_1 contains $F_W^*(A_0)$. Since A^* spans the knotted passage of W, the winding number of ∂D in V_1 is not zero. There is then a polyhedral solid torus V_2 in S^3 such that $\partial D \subset \operatorname{Int} V_2$, the winding number of ∂D in V_2 is not zero, and the core K of V_2 is an overhand knot. Therefore K is a companion knot of ∂D and genus $\partial D \geqq$ genus K **[4]**, **[10]**. This involves a contradiction since genus $\partial D = 0$ and genus $K > 0$.

PROPOSITION 3.3. *There is a dense subset of E^3 accessible at each point only by locally knotted arcs.*

PROOF. M is a dense subset of the open unit ball U. $f(M)$ is the desired set where $f(v) = (1 - |v|)^{-1} \cdot v$ for $v \in U$.

4. Related positional questions.

(1) Is there a set in E^3 accessible at each point by tame arcs which can be embedded as a set accessible at each point only by wild arcs?

(2) Is there such a set which can also be embedded as a set accessible at each point only by tame arcs?

The following may be of interest in connection with question 2.

EXAMPLE 2. *A connected locally connected subset T in the interior of a 3-cell I^3 such that each point of T is accessible only by tame arcs.* Let C_i be a subdivision of I^3 into 2^{3i} small cubes. Then $T = \operatorname{int} I^3 - \bigcup X_i$ where the X_i's are defined as follows. X_1 is the center of I^3. X_2 is the sum of 2^3 straight line segments from X_1 to the center of the 2^3 cubes in C_1. To define X_{i+1}, we consider a cube C of C_i. Then $X_i \cap C$ is either a vertex of C or a diagonal of C. If it is a diagonal, $X_{i+1} \cap C$ is this diagonal and $X_{i+1} = X_{i+1} \cap C = X_i \cap C$. If it is a vertex, then $X_{i+1} \cap C$ is the closed segment from this vertex to the center of C.

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