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H-SPACES WHICH ARE CO-H-SPACES

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ABSTRACT. It is shown that a space, having the homotopy type of a CW complex of finite type, admitting both H-space and co-H-space structures must have the homotopy type of a point or an n-sphere for $n=1$, 3 or 7.

Various people have considered spaces which admit both H-space and co-H-space structures. For example C. S. Hoo [5] showed that the set of homotopy classes of maps from a polyhedron to such a space forms a Moufang loop thereby extending results of C. W. Norman and R. O'Neill. On the other hand Curtis and Dugundji [2] showed that if a compact Lie group admits a H-cogroup structure then it has rank 1. Here we offer the following result.

THEOREM. *Let X have the homotopy type of a connected CW complex of finite type (of possibly infinite dimension). If X admits both an H-space and co-H-space structure then X has the homotopy type of S^1, S^3, S^7 or a point.*

REMARK. Adams and Walker [1] have exhibited a four-dimensional countably infinite CW complex T which is surprisingly both an Eilenberg-Mac Lane space of type $(Q, 3)$ and a Moore space of the same type. Since T is a suspension by its construction, we see that the hypothesis about finite type is essential to the theorem.

To prove the theorem let us suppose that X is not contractible. This implies that $\tilde{H}^*(X)\neq 0$ (or else $\tilde{H}_*(X)=0$ and the Whitehead Theorem would imply contractibility, since $\pi_1(X)\cong H_1(X)$ for H-spaces). Our first step is

CLAIM 1. *If R is the field Q or Z_p, p a prime, the Hopf algebra $H^*(X; R)$ is the exterior algebra on a single (necessarily primitive) generator of odd degree n.*

PROOF. We first observe that all cup-products of elements of positive degree in $H^*(X; R)$ vanish. To verify this folklore result note that by

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hypothesis the diagonal map $\Delta: X \to X \times X$ factors up to homotopy through $X \vee X$; since Δ^* gives cup-product it therefore suffices to prove that $i^*: \tilde{H}^*(X \times X) \to \tilde{H}^*(X \vee X)$, for any coefficient group, is zero. Identifying $\tilde{H}^*(X \vee X)$ with $\tilde{H}^*(X) \oplus \tilde{H}^*(X)$, naturality implies that

$$i^*(\alpha \times \beta) = i^*(\alpha \times 1) \cup i^*(1 \times \beta) = (\alpha + 0) \cup (0 + \beta) = 0 + 0.$$

Now suppose that $R = \boldsymbol{Q}$. Because of our assumptions on X we can apply the Leray Structure Theorem, [**8**, p. 268] or [**6**], to find that the algebra $H^*(X; \boldsymbol{Q})$ is the tensor product of a polynomial algebra and an exterior algebra on generators of odd degree. Claim 1 then follows from the above observation. A similar argument based on the Borel Structure Theorem takes care of the $R = \boldsymbol{Z}_p$ case.

COMMENT. The Hopf algebra $H_*(T) \otimes \boldsymbol{Q}$ shows that the Leray Structure Theorem is false without the finite type assumption; here, T is the space mentioned in the Remark.

CLAIM 2. *For any coefficient group* G, $H^*(X; G) \cong H^*(S^n; G)$.

PROOF. By Claim 1 this is true for $G = \boldsymbol{Q}$ or $\boldsymbol{Z}_p$, p prime. Now the homology of X is of finite type so we have an exact sequence, [**8**, p. 246],

$$0 \to H^q(X) \otimes G \to H^q(X; G) \to \mathrm{Tor}(H^{q+1}(X), G) \to 0.$$

As a consequence, we see that Claim 2 will follow if we show that the $G = \boldsymbol{Z}$ case is true since all Tor terms will be zero.

First, if $q \neq 0, n$ then $H^q(X) \otimes \boldsymbol{Q} \cong H^q(X; \boldsymbol{Q}) = 0$ implies that $H^q(X)$ is a finite group. If $\boldsymbol{Z}_{p^\alpha}$ is a direct summand we then have

$$\boldsymbol{Z}_p \cong \boldsymbol{Z}_{p^\alpha} \otimes \boldsymbol{Z}_p \subset H^q(X) \otimes \boldsymbol{Z}_p \cong H^q(X; \boldsymbol{Z}_p) = 0;$$

consequently, $H^q(X) = 0$. The same method also shows that $H^n(X) \cong \boldsymbol{Z}$ so the proof of Claim 2 is complete.

CLAIM 3. $X \simeq S^n$.

PROOF. S^n is q-simple for all q (as S^1 is an H-space) and Claim 2 implies that $H^q(X, *; \pi_q S^n) = 0 = H^{q+1}(X, *; \pi_q S^n)$ for all $q \geq n+1$. From obstruction theory [**9**, pp. 73, 78] we obtain a bijective correspondence between $[X, S^n]$ and $H^n(X; \pi_n S^n) \cong \boldsymbol{Z}$ associating $[g]$ with $g^*\mu$ where $\mu \in H^n(S^n; \pi_n S^n) \cong \boldsymbol{Z}$ is the fundamental class. Letting $f: X \to S^n$ be associated with a generator, we conclude from Claim 2 that f^* is an isomorphism of integral cohomology since $f^*\mu$ generates in degree n. By naturality of the Ext-Hom sequence it follows that $f_*: H_*(X) \cong H_*(S^n)$.

Just as in our initial remark concerning contractibility, we observe that X is simply connected iff $n > 1$. In this case the Whitehead Theorems imply that f is a homotopy equivalence. The case $n = 1$ requires more work.

First, by naturality of the Hurewicz homomorphism, $f_*:\pi_1(X)\cong\pi_1(S^1)$. Next, let $\bar{X}$ be the universal covering space of X. Then $\pi_1(X)$ operates trivially on $\tilde{H}_*(\bar{X})$ by [**7**, p. 479] so there exists a exact sequence

$$0 \to H_{q-1}(\bar{X}) \to H_q(X) \to H_q(\bar{X}) \to 0$$

for each q by [**4**, p. 467]. Hence, $\tilde{H}_*(\bar{X})=0$. Letting $\bar{f}\colon \bar{X}\to \boldsymbol{R}$ be the induced universal covering morphism it follows that $\bar{f}_*: H_*(\bar{X})\cong H_*(\boldsymbol{R})$. It follows from a form of Whitehead's Theorem [**3**, p. 113] that f is a homotopy equivalence. Hence, Claim 3 is established.

The proof of our theorem is completed by observing that since X is an H-space so must S^n and therefore $n=1$, 3 or 7.

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