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AN EMBEDDING THEOREM FOR COMMUTATIVE LATTICE-ORDERED DOMAINS

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ABSTRACT. In a recent paper Conrad and Dauns have shown that a finitely-rooted lattice-ordered field R, in which multiplication by a positive special element is a lattice homomorphism, can be embedded in a formal power series l-field with real coefficients, provided that the value group of R is torsion-free. In this note it is shown that their theorem is true when R is a commutative integral domain.

The proof of this theorem will be the same as the proof of Conrad and Dauns' result [**3**, Theorem III] once it has been established that a quotient ring of R can be lattice-ordered.[1] Their arguments do not use that R is a field but only that the set of positive special elements of R is a multiplicative group.

This paper represents a portion of the author's dissertation written at the University of Illinois at Urbana-Champaign under the direction of Professor Elliot C. Weinberg.

1. **Extending the lattice order to a quotient ring.** The following lemma may be thought of as a generalization of [**3**, Theorem I] (as may Lemma 2, also).

LEMMA 1. *Let R be an l-ring containing a positive invertible element u. Then right (left) multiplication by u is a lattice homomorphism (equivalently, an l-automorphism of the underlying l-group of R) if and only if u^{-1} is positive.*

PROOF. If $x \to xu$ is a lattice homomorphism, then $(1 \vee 0)u = u \vee 0 = u$; so $1 > 0$. Also, $(u^{-1} \vee 0)u = 1 \vee 0 = 1$ implies $u^{-1} \vee 0 = u^{-1}$; i.e., $u^{-1} > 0$. Conversely, suppose that $u^{-1} \in R^+$. If a is any positive element of R, then $xa \vee ya \leqq (x \vee y)a$. So,

$$x \vee y = xuu^{-1} \vee yuu^{-1} \leqq (xu \vee yu)u^{-1} \leqq (x \vee y)uu^{-1} = x \vee y.$$

Thus $(x \vee y)u = xu \vee yu$.

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[1] The author would like to thank Professor Paul Conrad for pointing out a mistake in an earlier version of Theorem 2.

PROPOSITION 1. *Let R be an l-ring, and let Σ be a multiplicative subset of positive regular elements. Suppose that R is a right Ore ring with respect to Σ, and let Q be its classical right quotient ring with respect to Σ. Then Q can be made into an l-ring extension of R, in which the inverse of each element of Σ is positive, exactly when left multiplication and right multiplication by each element of Σ are lattice homomorphisms. If this is the case, the lattice order of Q is uniquely determined by that of R.*

PROOF. Suppose that $x^+a=(xa)^+$ and $ax^+=(ax)^+$ for all $x\in R$, $a\in\Sigma$. Define $Q^+=\{q\in Q:q$ has a representation of the form $q=xa^{-1}$ for some $(x,a)\in R^+\times\Sigma\}$. We claim that (Q, Q^+) is an l-ring extension of (R, R^+) and that $a\in\Sigma$ implies $a^{-1}\in Q^+$. We first note that if $q\in Q^+$ and $q=yb^{-1}$, then $y\in R^+$. Suppose, then, that $xa^{-1}=yb^{-1}$ with $x\in R^+$. There exists $(c,d)\in R\times\Sigma$ such that $xc=yd$ and $ac=bd$. Then $0=(ac)^-=ac^-$ implies that $c\in R^+$, since a is regular. Therefore $0=(xc)^-=(yd)^-=y^-d$, so $y\in R^+$.

It is easily seen that Q^+ is a positive cone for the ring Q and that $Q^+\cap R=R^+$. Note that the least upper bound of two elements in R is also their least upper bound in Q. For suppose that $(q,x)\in Q\times R$, $q=ya^{-1}$, and $q\geqq\{x,0\}$. Then $y\geqq\{xa,0\}$; so $y\geqq(xa)^+=x^+a$ and $q\geqq x^+$. Note, also, that if $a\in\Sigma$, then $a^{-1}=a(a^2)^{-1}\in Q^+$.

If $q=xa^{-1}\in Q$, define $q^*=x^+a^{-1}$. Then q^* is independent of the representation of q. For if $q=yb^{-1}$, then there exists $(c,d)\in R\times\Sigma$ such that $bc=ad$ and $yc=xd$. Since a, b, and d are in Σ, $c\in R^+$ and $c^{-1}\in Q^+$. Since right multiplication by c is an order isomorphism of Q onto Q, and thus preserves all existing sups, $y^+c=(yc)^+=(xd)^+=x^+d$. Thus $y^+b^{-1}=x^+a^{-1}$, and q^* is well defined. It is now easily seen that $q^*=q\vee 0$.

The converse is immediate from Lemma 1. For uniqueness, suppose that (Q,P) is an l-ring extension of (R,R^+) such that $a\in\Sigma$ implies $a^{-1}\in P$. Then, clearly, $Q^+\subseteq P$. If $xa^{-1}\in P$, then $x=(xa^{-1})a\in P\cap R=R^+$. So $P=R^+$.

We have essentially used Anderson's proof of the special case of the following corollary, i.e., that in which Σ is the set of all regular elements of the unital f-ring R [**1**, Theorem 5.1], to prove Proposition 1.

COROLLARY. *Let R be a right Ore ring with respect to a multiplicative subset of regular elements Σ, and let Q be its classical right quotient ring with respect to Σ. If R is an f-ring, then Q can be made into an f-ring extension of R in exactly one way.*

PROOF. Let $\Sigma^+=\Sigma\cap R^+$. If $q\in Q$, then $q=xa^{-1}=xa(a^2)^{-1}$ and $a^2\in\Sigma^+$. Thus Q is the right quotient ring of R with respect to Σ^+. Now use Proposition 1.

2. *D*-**domains.** A *value* of a nonzero element g in an l-group is a convex l-subgroup that is maximal with respect to the exclusion of g. A *special element* of an l-group G is an element that has exactly one value. Note that a special element is comparable to 0. G is *finitely-valued* if each of its elements has only a finite number of values. For the remainder of this paper, by a special element we shall mean a positive special element. A *basic element* is a positive nonzero element g for which the convex l-subgroup generated by g, $C(g)$, is totally ordered. It is known that a basic element is special. For the theory of special elements see [**2**]. The following lemma is an immediate consequence of Lemma 1.

LEMMA 2. *Let R be an l-ring containing an invertible element u such that u and u^{-1} are both positive.*
(a) *If $g\in R$, then g is special (basic) if and only if ug is special (basic).*
(b) *1 is special (basic) if and only if u is special (basic).*
(c) *u is special (basic) if and only if u^{-1} is special (basic).*

An *l-domain* is an l-ring R in which $R^+\backslash\{0\}$ is a multiplicatively closed subset [**6**].

COROLLARY. *If R is an l-domain, then u is basic.*

PROOF. $C(1)$, the convex l-subgroup of R generated by 1, is an f-ring since 1 is a strong order unit in $C(1)$. Since $C(1)$ is an l-domain, it is totally ordered.

This corollary is actually a generalization of the fact that a lattice-ordered division ring in which the inverse of every positive element is positive must be totally ordered. This is proven in [7, p. 199] for the commutative case.

By a *D-domain* we shall mean a commutative lattice-ordered ring R, without zero divisors, such that
(a) the set S of special elements of R is nonempty, and
(b) multiplication by an element of S is a lattice homomorphism.

PROPOSITION 2. *Let R be a commutative lattice-ordered integral domain. Then R is a D-domain if and only if*

$$S_1 = \{0 \neq s \in R^+ : \textit{multiplication by } s \textit{ is a lattice homomorphism}\}$$

is nonempty and $S=S_1$. In particular, the special elements in a D-domain form a multiplicatively closed subset.

PROOF. If R is a a D-domain, then $S\subseteq S_1$, so S_1 is nonempty. Clearly S_1 is multiplicatively closed. Let Q be the ring of quotients of R with respect to S_1. By Proposition 1, Q is an l-ring extension of R, and its positive cone is given by

$$Q^+ = \{as^{-1} \in Q : a \in R^+, s \in S\}.$$

If $s \in S_1$, then s and s^{-1} are in Q^+. By the previous corollary, s is basic in Q; hence it is also basic in R. Thus $S = S_1$. The converse is trivial

In [**3**, Theorem I] it is shown that an l-field R is a D-domain if and only if its set of special elements S forms a multiplicative group. The following example shows that a finitely-rooted, commutative, lattice-ordered domain, in which S is multiplicatively closed, need not be a D-domain.

EXAMPLE 1. Let $R = F[x]$ be the polynomial ring over the totally ordered field F, and let $R^+ = \{\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n : \alpha_0 \geqq 0 \text{ and } \alpha_n \geqq 0\}$. As an l-group R is the direct sum of two totally ordered groups, and so it has only two roots. Its set of special elements is

$$S = \{\alpha : 0 < \alpha \in F\} \cup \{\alpha_1 x + \cdots + \alpha_n x^n : \alpha_n > 0\}.$$

Since $1 \wedge x = 0$, but $x \wedge x^2 > 0$, R is not a D-domain.

For the remainder of this paper let R be a D-domain with special elements S. By Proposition 2, S is a multiplicatively closed subset of R. If Q is the ring of quotients of R with respect to S, then Proposition 1 says that Q is an l-ring extension of R. Let T be the set of special elements of Q.

PROPOSITION 3. *T is the quotient po-group of S; i.e., $T = \{as^{-1} \in Q : a, s \in S\}$.*

PROOF. If $a, s \in S$, then $(as^{-1})^{-1} = sa^{-1} \in Q^+$. So $as^{-1} \in T$, by the corollary to Lemma 2. Conversely, suppose that $as^{-1} \in T$. By Lemma 2(a), $a \in T$. Let $Q(a)$ (respectively $R(a)$) be the convex l-subgroup of Q (respectively R) generated by a. Then $Q(a) = \text{lex } N$ for a proper convex l-subgroup N of $Q(a)$ [**2**, Theorem 3.6]. But then $N \cap R(a) \subsetneq R(a)$, and $R(a) = \text{lex}(N \cap R(a))$. Thus $a \in S$.

COROLLARY 1. *Q is a D-domain and $T = \{q \in Q^+ : q^{-1} \in Q^+\}$.*

PROOF. This follows immediately from Proposition 3 and the previous corollary.

Note that $T \cap R = S$ and that S is actually the set of basic elements of R.

COROLLARY 2. *R is finitely-valued if and only if its underlying l-group is a direct sum of totally ordered groups.*

PROOF. The only if part follows from the fact that each special element is basic and from [**8**, Theorem 2.12]. The if part is trivial.

PROPOSITION 4. (a) *If R is finitely-valued, then so is Q.*

(b) *R has exactly n roots if and only if Q has exactly n roots.*

PROOF. Suppose that R is finitely-valued and $gs^{-1} \in Q^+$. Then g is the sum of pairwise disjoint special elements of R; $g=g_1+\cdots+g_n$ [**2**, Theorem 3.7]. Thus gs^{-1} is the sum of pairwise disjoint special elements of Q. So Q is finitely-valued.

As is well known R has exactly n roots if and only if R has a basis containing exactly n elements [**5**]. Suppose that R has exactly n roots and $\{g_i u_i^{-1} : i=1, \cdots, m\}$ is a set of pairwise disjoint elements of Q. Let $s_i = u_1 \cdots u_{i-1} u_{i+1} \cdots u_m$. Then $\{g_i s_i : i=1, \cdots, m\}$ is a set of pairwise disjoint elements of R. So $m \leqq n$, and Q has at most n roots. But R contains n disjoint elements, so Q has at least n roots. Thus Q has exactly n roots.

Conversely, if $\{g_i s^{-1} : i=1, \cdots, n\}$ is a basis of Q, then $\{g_i : i=1, \cdots, n\} \subset S$. Since R cannot have more than n disjoint elements, $\{g_i : i=1, \cdots, n\}$ is a basis of R.

PROPOSITION 5. *Each D-domain contains a unique largest convex l-subring that is a finitely-valued D-domain.*

PROOF. We first show that the sum of a finitely-rooted convex l-subgroup A and a totally ordered subgroup B of an l-group G is a finitely-rooted l-subgroup. For $A+B$ is an l-subgroup of G since $B+A/A$ is one of G/A. Note that since $B+A/A \cong B/A \cap B$, A is a prime subgroup of $B+A$. Suppose that A has only n roots, and let $\{a_i+b_i : a_i \in A, b_i \in B; i=1, \cdots, m\}$ be m disjoint elements in $A+B$. If $b_i \in A$ for all i, then clearly $m \leqq n$. If $b_i \notin A$ for some i, then $a_j+b_j \in A$ for $j \neq i$, and so $m \leqq n+1$. Thus $A+B$ is finitely-rooted. By induction, it is easily seen that the sum of n totally ordered convex l-subgroups of an l-group has at most n roots.

Now let S be the set of special elements of the D-domain R, and let A be the additive subgroup of R generated by S:

$$A = \{g_1 + \cdots + g_n : |g_i| \in S\}.$$

Since each special element is basic, $g \in S$ implies $C(g) \subseteq A$. So, by the preceding paragraph, A is a convex l-subring of R. Since each special element of A is special in R [**2**, Theorem 3.5], A is a D-domain. It is clearly the largest convex l-sub-D-domain of R that is finitely-valued.

Note that Example 2 and the remarks following it show that the finitely-valued part of a D-field need not be a field.

If Γ is the value set of R, then the mapping $v_R : S \to \Gamma$ that sends each element of S to its value is order preserving. Moreover, v_R has the following properties:

(1) $v_R(s)=v_R(t)$ if and only if $C(s)=C(t)$.

(2) $v_R(s)<v_R(t)$ if and only if s is infinitely smaller than t.

(3) If $v_R(s)=v_R(t)$ and $a \in S$, then $v_R(as)=v_R(at)$.

The last property implies that the multiplication in S can be transferred to $v_R(S)$ via v_R. Thus, using additive notation, $v_R(S)$ becomes a rooted partially-ordered semigroup if addition is defined by $v_R(s)+v_R(t)=v_R(st)$. We will say that $v_R(S)$ is torsion-free if for all α, $\beta \in v_R(S)$ and $0 \neq n \in Z^+$, $n\alpha = n\beta$ implies $\alpha=\beta$.

We can, of course, do the same thing for Q. In this case $v_Q(T)$ becomes a rooted *po*-group since T is a group. It is clear that $v_Q(T)$ is the quotient *po*-group of $v_R(S)$, where $v_R(S)$ is embedded in $v_Q(T)$ via $v_R(a) \to v_Q(a)$. Note that $v_Q(T)$ is a torsion-free group exactly when $v_R(S)$ is a torsion-free semigroup. For brevity, we will call $(Q, T, v_Q(T))$ the *quotient system* of $(R, S, v_R(S))$.

Now suppose that R is finitely-valued. Then $v_R(S)=\Gamma$ [**2**, Theorem 3.8]. By Proposition 4, Q is also finitely-valued; so $v_Q(T)=\Delta$ is the value set of Q. We now state the main results of this paper. As mentioned earlier, their proofs are identical with those given in [**3**] for the case that R is a field. $V(\Gamma, \boldsymbol{R})$ is the formal power series l-ring with exponents in Γ, coefficients in the real field $\boldsymbol{R}$, and whose lattice order is that of its underlying Hahn product.

THEOREM 1. *Let R be a finitely-valued D-domain whose value semigroup Γ is torsion-free. Then the lattice order of R can be extended to a total* (*ring*) *order of R.*

THEOREM 2. *Let R be a finitely-rooted D-domain, and let* (Q, T, Δ) *be the quotient system of* (R, S, Γ). *If Γ is torsion-free there is a value preserving l-isomorphism of Q into the l-field* $V(\Delta, \boldsymbol{R})$, *whose restriction to R is also value preserving.*

3. **Remarks.** The following example shows that the quotient l-ring Q of a finitely-valued D-domain R (for which Γ is torsion-free) need not be its quotient field. Note that any such Q must be l-simple, i.e., 0 and Q are its only l-ideals. If Q is finitely-rooted we do not know if it must be a field.

EXAMPLE 2. Let $R=F[x]$ be the polynomial ring over the totally ordered field F. R becomes a D-domain if its positive cone is defined by $R^+=\{\sum_{i=0}^n \alpha_i x^i : \alpha_i \geqq 0 \text{ for each } i\}$. Then $S=\{\alpha x^n : \alpha > 0\}$, $Q=\{\sum_{i=-m}^n \alpha_i x^i\}$, $Q^+=\{\sum_{i=-m}^n \alpha_i x^i : \alpha_i \geqq 0\}$, and $T=\{\alpha x^n : n \in Z \text{ and } \alpha > 0\}$. Using (1) and (2) it is easily seen that $\Gamma \cong v(F) \oplus Z^+$ and $\Delta \cong v(F) \oplus Z$, where $v(F)$ is the value group of F and $Z(Z^+)$ is the trivially ordered group (semigroup) of (positive) integers.

We have not been able to determine whether the quotient field E of a D-domain R can be made into an l-ring extension of R. In the preceding example Q can be value-embedded in an l-field G that is a D-domain, but which is not finitely-valued:

$$G = \left\{ \sum_{i=-\infty}^{n} \alpha_i x^i : \alpha_i \in F \text{ and } n \in Z \right\}$$

and

$$G^+ = \left\{ \sum_{i=-\infty}^{n} \alpha_i x^i : \alpha_i \geqq 0 \text{ for each } i \right\}.$$

If R is finitely-rooted and Δ is torsion-free, and if E is an l-subring of $V(\Delta, \boldsymbol{R})$, then E is a D-domain.

Examples of D-domains may be obtained as follows. Let Δ be a finitely-rooted torsion-free abelian *po*-group, and let Δ_1 be the unique totally ordered group whose underlying group is Δ and such that $\Delta^+ \subseteq \Delta_1^+$ [**3**, p. 387]. Let $V = V(\Delta, \boldsymbol{R})$ and $V_1 = V(\Delta_1, \boldsymbol{R})$. Then V and V_1 are the same ring, and $V^+ \subseteq V_1^+$ [**3**, 2.2]. Let

$$\begin{aligned} R &= \{g \in V_1 : \text{value of } g \leqq 0\} \cup \{0\} \\ &= \{g \in V_1 : |g| \leqq n \text{ for some positive integer } n\}. \end{aligned}$$

Then R is a sub-D-domain of V, and $Q(R) = V$. If Δ is not abelian, then R is a *noncommutative D-domain.*

Finally, we note that the results of §2 hold if R is only a commutative l-domain (but otherwise satisfying the defining properties of a D-domain). For the proof of Proposition 2 remains valid provided S_1 consists of regular elements. But if $a \in S_1$, then $ab = 0$ implies $ab^+ = ab^- = 0$. Hence $b = 0$, and a is regular. Now Propositions 3 through 5 hold for R; and the proof of Theorem 1 shows that if R is finitely-valued and Γ is torsion-free, then R is, in fact, a domain.

ADDED IN PROOF. Professor Henriksen has informed us that the quotient field of a finitely-valued D-domain R cannot always be made into an l-ring extension of R.

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