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DIRICHLET L-FUNCTIONS AND PRIMITIVE CHARACTERS

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ABSTRACT. It is well known that a Dirichlet L-function $L(s, \chi)$ has a functional equation if the character χ is primitive. This note proves the converse result. That is, if $L(s, \chi)$ satisfies the usual functional equation then χ is primitive.

1. **Introduction.** For a positive integer k, let χ be any character modulo k, let $L(s, \chi)$ denote the L-function defined for $R(s)>1$ by the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

let $G(n, k)$ denote the Gauss sum

$$G(n, k) = \sum_{h=1}^{k-1} \chi(h) e^{2\pi i n h/k},$$

and let $G(\chi)=G(1, \chi)$. It is well known that if χ is *primitive* then $L(s, \chi)$ satisfies the functional equation

$$(1) \quad L(1-s, \chi) = (2\pi)^{-s}\Gamma(s)k^{s-1}\{e^{-is\pi/2} + \chi(-1)e^{is\pi/2}\}G(\chi)L(s, \bar{\chi}).$$

A recent proof is given in **[1]**. This paper proves the converse result.

THEOREM 1. *If χ is a character modulo k and if $L(s, \chi)$ satisfies the unctional equation* (1), *then χ is a primitive character modulo k.*

The proof is based on two lemmas, each of which gives a necessary and sufficient condition for a character modulo k to be primitive.

2. **Lemmas.** The first lemma is a restatement of Theorem 1 in **[2]**.

LEMMA 1. *A character χ modulo k is primitive if, and only if,*

$$G(n, \chi) = \bar{\chi}(n)G(\chi)$$

for every integer n.

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The next lemma involves the function $F(x, s)$ defined for each real x as the analytic continuation of the Dirichlet series

$$F(x, s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \qquad R(s) > 1.$$

This function was used recently in **[1]** to give a new representation of $L(s, \chi)$ for primitive characters.

LEMMA 2. *For any character χ modulo k and any complex s, let*

$$L^*(s, \chi) = \sum_{h=1}^{k-1} \chi(h) F\left(\frac{h}{k}, s\right).$$

Then we have

$$L^*(s, \chi) = G(\chi) L(s, \bar{\chi}) \tag{2}$$

for all s if and only if χ is primitive.

PROOF. If $R(s) > 1$ we have

$$L^*(s, \chi) = \sum_{h=1}^{k-1} \chi(h) \sum_{n=1}^{\infty} n^{-s} e^{2\pi i n h/k} = \sum_{n=1}^{\infty} G(n, \chi) n^{-s} \tag{3}$$

and

$$G(\chi) L(s, \bar{\chi}) = \sum_{n=1}^{\infty} G(\chi) \bar{\chi}(n) n^{-s}. \tag{4}$$

If (2) holds for all s then it also holds for $R(s) > 1$ and the two Dirichlet series in (3) and (4) have the same coefficients. By Lemma 1 it follows that χ is primitive.

Conversely, if χ is primitive, Lemma 1 shows that the two functions in (3) and (4) are equal for $R(s) > 1$ and hence they must be equal for all s.

3. **Proof of Theorem 1.** We refer to equation (39) in **[1]** and note that it is valid for every character χ modulo k. This gives us the relation

$$L(1 - s, \chi) = f(s, \chi) L^*(s, \chi) \tag{5}$$

where

$$f(s, \chi) = (2\pi)^{-s} \Gamma(s) k^{s-1} \{e^{-is\pi/2} + \chi(-1) e^{is\pi/2}\}.$$

If $L(s, \chi)$ satisfies the functional equation (1) we also have

$$L(1 - s, \chi) = f(s, \chi) G(\chi) L(s, \bar{\chi}). \tag{6}$$

From (5) and (6) we find $G(\chi) L(s, \bar{\chi}) = L^*(s, \chi)$ for all s. Therefore, by Lemma 2, χ is primitive.

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