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DUAL NECKLACES OF n-DIMENSIONAL CUBES

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ABSTRACT. It has been shown that certain finite configurations, called dual necklaces, of euclidean spheres yield information on longest rectilinear circuits on the sphere centers. In the present paper, the existence of dual necklaces of k n-dimensional cubes is discussed and the values of k are determined. A more general configuration of cubes, called a multidual necklace, is treated similarly.

I. **Introduction.** The object of this paper is to investigate the existence of geometric realizations of two extremal conditions. These conditions, given below in Definition 1, have their origin in the following "dual necklace theorem" of D. Sanders [**1**]: A rectilinear polygon on r vertices in the (euclidean) space E^n, $n>1$, is longest in its covertex class if a (closed) sphere may be centered on each vertex such that each sphere intersects all others except for the two centered on the two adjacent vertices given by the polygon.

DEFINITION 1. A *dual necklace* is a collection of k ($k\geqq 3$) subsets of coordinate space R^n such that (condition 1) each set intersects all but two others in the collection, and (condition 2) when a unique point is designated in each set as "center" and the centers of nonintersecting sets joined by straight line segments, a rectilinear polygon is formed.

It has been shown by Warren Becker that dual necklaces of k spheres exist in E^n, $n>1$, for all $k\geqq 3$. Define the metric space $S^n=(R^n, s)$ by $s(x, y)=\max_{1\leqq i\leqq n} |x_i-y_i|$; the metric spheres of the space S^n are of course n-dimensional cubes.

II. THEOREM 1. *Dual necklaces of k (closed) spheres in S^n exist for precisely $3\leqq k\leqq 2n+3$ if $n\geqq 3$, and for $3\leqq k\leqq 6$ if $n=2$.*

DEFINITION 2. *A multi-dual necklace* is a collection of subsets of R^n satisfying condition 1 of Definition 1.

THEOREM 2. *There exist multi-dual necklaces of k closed spheres in the space S^n for precisely $3\leqq k\leqq 4n$.*

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LEMMA 1. *There does not exist a dual necklace of k rectangular parallelopipeds (n-dimensional, with edges parallel to the coordinate axes) if* $k>2n+3$, *where* $n\geqq 3$, *or for* $k>6$, *if* $n=2$.

PROOF. Suppose there were such a collection, C. Let P_0 belong to C and let P_1 and Q_1 be the two members of C which do not intersect P_0. Then P_1 intersects Q_1, or else we would be forced to join the centers of P_0, P_1, Q_1 to obtain a triangle. This is impossible since $n\geqq 2$. Moreover, there are additional members P_2 and Q_2 of C such that $P_2\cap Q_1=\varnothing$, $Q_2\cap P_1=\varnothing$, but $P_2\cap P_1\neq\varnothing$, $Q_2\cap Q_1\neq\varnothing$ and $P_2\cap Q_2\neq\varnothing$. If this were not the case, that is, if there were a member S in C which intersected P_0 but neither P_1 nor Q_1 we must join the centers of P_0, P_1, S, Q_1, and P_0 in that order, obtaining a quadrilateral. Similarly, P_2 and Q_2 must intersect or we obtain a pentagon. Finally, each of the remaining members of C must intersect P_0, P_1, and Q_1.

Let us continue by defining P_0 to be a particular member of C. We will denote the perpendicular projection of a member R of C on th· ith coordinate axis by R^i, $i=1,\cdots,n$. Let P_0 be any member of C such that the right-hand endpoint of P_0^1 is minimal among all right-hand endpoints of the intervals R^1, for all R in C. Observe that two members P and Q in C intersect if and only if we have $P^i\cap Q^i\neq\varnothing$ for all $i=1,\cdots,n$. But $P_2\cap P_0\neq\varnothing$ and $Q_2\cap P_0\neq\varnothing$ implies $Q_2^1\cap P_0^1\neq\varnothing$ and $P_2^1\cap P_0^1\neq\varnothing$. Hence, we may assume, by symmetry, that $P_2^1\cap P_1^1\neq\varnothing$ and $P_2^1\cap Q_1^1\neq\varnothing$. But then $P_2\cap Q_1=\varnothing$ implies that $P_2^i\cap Q_1^i=\varnothing$ for some $i>1$; and, by symmetry, we may assume that $i=2$.

Moreover, $P_2^2\cap Q_2^2\neq\varnothing$ and $Q_2^2\cap Q_1^2\neq\varnothing$. But P_2^2 and Q_1^2 are separated by an open interval I_1. Hence, Q_2^2 contains I_1, since Q_2^2 is connected. In fact, if R belongs to C, $R^2\cap P_2^2\neq\varnothing$ and $R^2\cap Q_1^2\neq\varnothing$, then R contains I_1; so, $R^2\cap Q_2^2\neq\varnothing$. Further, if $R^1\cap P_0^1\neq\varnothing$, then $R^1\cap Q_2^1\neq\varnothing$, since both R^1 and Q_2^1 contain the right-hand endpoint of P_0^1.

Thus, we have shown that, if we are in the plane, any R in C intersecting P_0, P_2, and Q_1 must intersect Q_2 as well, contradicting the requirement that exactly two rectangles of C do not intersect Q_2.

In $n>2$ dimensions, we suppose that C has at least $2n+4$ members, the first five of which are P_0, P_1, Q_1, P_2, and Q_2 as above. We then iterate the above argument, obtaining a list of pairs of members of C: P_3, Q_3; P_4, Q_4; $\cdots$; P_{n+1}, Q_{n+1}, such that each P and Q of a pair intersect each other and all the preceding members of the list except the preceding Q or P respectively. Thus, Q_{n+1} intersects all others on the list except P_n. Further, we may assume, by symmetry that for each $i=2,\cdots,n-1$, P_{i+1}^{i+1} and Q_i^{i+1} are separated on the $(i+1)$th coordinate axis by an open interval I_i. Hence, and member Q_0 of C not already accounted for on the preceding list must intersect P_{n+1}. For, P^i_{n+1} contains the intervals I_i by virtue of P_{n+1} intersecting P_i and Q_{i-1}, $2 \leqq i \leqq n$; and Q_0 also has this property. Moreover, both P^1_{n+1} and Q^1_0 contain the right-hand endpoint of P^1_0. But this contradicts the requirement that P_{n+1} intersect all but two members of C, completing the proof. Q.E.D.

PROOF OF THEOREM 1. To show the existence of dual necklaces claimed in the theorem, proceed as follows: A realization of $2n+3$ (hypercubic) spheres in S^n, $n \geqq 3$, $P_0, P_1, Q_1, \cdots, P_{n+1}, Q_{n+1}$ is obtained by following the pattern discussed in the previous proof, except that we require P_{n+1} and Q_{n+1} to be disjoint. We need only give their projections on the n coordinate axes. (See Table I.)

The symbol "*" on an axis means that the projections of spheres not explicitly accounted for on that axis take position *. Observe that on the x_1-axis P_0 does not intersect Q_1 or P_1; P_2 does not intersect Q_1. On the x_n-axis, P_{n+1} does not intersect Q_{n+1} or Q_n; P_{n-1} does not intersect Q_n. Finally, on all the remaining x_i-axes, P_{i+1} does not intersect Q_{i+2}, or Q_i; P_{i-1} does not intersect Q_i. However, all other intersections do occur, thus yielding the required realizations of $2n+3$ members.

Similarly, Table II gives a realization of $2n+2$ spheres in the space S^n, $n \geqq 3$.

It is clear that a realization in one space gives realizations in all higher dimensional S^n. Thus, we will be finished when we give realizations of size six and seven in S^2 and S^3, respectively. For the latter, we take spheres with centers at $(-1, 8, 8)$, $(10, 2, 8)$, $(3, 5, 3)$, $(6, 8, 1)$, $(12, 6, 8)$, $(6, 12, 10)$, $(6, 8, 12)$. For the former, take spheres with centers at $(2, 9)$, $(5, 12)$, $(8, 14)$, $(12, 5)$, $(8, 2)$, $(15, 8)$. In each case let the spheres have radii equal to four. Q.E.D.

DEFINITION 3. A dual necklace of k spheres in S *removes axis* x_i *if* it contains spheres P and Q such that $P^i \cap Q^i = \varnothing$.

LEMMA 2. *Each dual necklace of k spheres in S*, $n \geqq 2$, *removes at least r axes, where* $2r+1 \leqq k \leqq 2r+3$.

PROOF. Immediate by the proof of Lemma 1.

LEMMA 3. *There does not exist a collection of k rectangular parallelopipeds (n-dimensional, with edges parallel to the coordinate axes) satisfying condition* 1 *for* $k > 4n$ *in* R^n, $n \geqq 1$.

PROOF. We will call the members of such collections "spheres" for the sake of brevity. Now, each multi-dual necklace of spheres (abbreviated "mdn") may be conceived as a finite set of dual necklaces (abbreviated "dn") such that each sphere of one dn intersects every sphere in any other

TABLE I

$P_0 \quad P_2 \quad Q_2 \quad * \quad P_0 \quad P_1 \quad P_2\, Q_1 \quad Q_2 \quad * \quad P_1 \quad Q_1 \longrightarrow X_1$

$P_3 \quad P_1 \quad Q_1 \quad P_2 \quad Q_3\, P_4 \quad * \quad P_3 \quad Q_4 \quad P_1 \quad Q_2 \quad Q_1 \quad P_2\, Q_3 \quad P_4 \quad * \quad Q_4 \quad Q_2 \longrightarrow X_2$

$P_4 \quad P_2 \quad Q_2 \quad P_3 \quad Q_4\, P_5 \quad * \quad P_4 \quad Q_5 \quad P_2 \quad Q_3 \quad Q_2 \quad P_3\, Q_4 \quad P_5 \quad * \quad Q_5 \quad Q_3 \longrightarrow X_3$

(raise each index by one each time)

$P_n \quad P_{n-2}\, Q_{n-2}\, P_{n-1}\, Q_n\, P_{n+1} \quad * \quad P_n \quad Q_{n+1}\, P_{n-2} \quad Q_{n-1}\, Q_{n-2}\, P_n\, Q_{n-1}\, P_{n+1} \quad * \quad Q_{n+1} \quad Q_n \longrightarrow X_{n-1}$

$P_{n+1}\, P_{n-1} \quad P_n \quad * \quad Q_{n-1}\, P_{n+1} \quad Q_{n-1}\, P_{n-1}\, Q_n \quad P_n \quad * \quad Q_{n-1}\, Q_{n+1} \quad Q_n \longrightarrow X_n$

TABLE II

$P_0 \quad P_2 \quad Q_2 \quad * \quad P_0 \quad P_1 \quad P_2\, Q_1\, Q_2 \quad * \quad P_1 \quad Q_1 \longrightarrow X_1$

$P_1 \quad P_3\, Q_1 \quad P_2 \quad Q_3 \quad * \quad P_1\, P_3 \quad Q_2\, Q_1 \quad P_2 \quad Q_3 \quad * \quad Q_2 \longrightarrow X_2$

(raise each index by one each time)

$Q_0 \quad P_{n-1} \quad Q_{n-1} \quad P_n \quad * \quad Q_0\, P_{n-1} \quad Q_n\, Q_{n-1} \quad P_n \quad * \quad Q_n \longrightarrow X_n$

dn in the set. This is seen by considering the abstract graph formed by connecting spheres which do not intersect and obtaining thereby a finite set of disjoint cycles. Each sphere will be in a finite cycle, by the proof of Lemma 1. Now, letting $[k]$ denote a dn of $k \geqq 3$ spheres, a set of $j \geqq 1$ distinct dn's forming an mdn will be denoted by $[k_1, k_2, \cdots, k_j]$. We claim that there do not exist mdn's of types $[k_1, \cdots, k_j]$ in S^n for $j > n$. For, two different dn's in an mdn cannot remove the same axis; and each dn removes at least one axis. Hence, any mdn $[k_1, \cdots, k_j]$ removes at least j different axes.

Next, we claim that there is an mdn of type $[k_1, \cdots, k_n]$, where $k_i = 4$ for each $i = 1, \cdots, n$ in S^n. Moreover, $4n > k'_1 + \cdots + k'_j$ where $[k'_1, \cdots, k'_j]$ is any other mdn of the kind under consideration.

For the existence, let the k be defined by the following four spheres, each having a radius of four; the ith coordinate of their centers is 0, except for the $i = j$th, where the coordinates are -6, -5, 5, and 6. Intersection properties are easily verified on the coordinate axes.

Secondly, let $[k_1, \cdots, k_j]$ be an mdn of type other than $[4_1, \cdots, 4_n]$ in S^n. Hence, some $k_i \neq 4$, say k_j, or $j < n$. Consider $[k_1, \cdots, k_{j-1}]$, $j > 1$. (The case $j = 1$ is trivial.) If $2k+1 \leqq k_j \leqq 2k+3$, then $[k_j]$ removed at least k axes, say $x_1, \cdots, x_k$, upon which the projections of the remaining spheres all intersect. That is, on these axes there are intervals $I_1, \cdots, I_k$ which the projections of every sphere of $[k_1, \cdots, k_{j-1}]$ contain. On the remaining axes $x_{k+1}, \cdots, x_n$ the projections of any sphere P of $[k_j]$ intersects the projections of every sphere of $[k_1, \cdots, k_{j-1}]$. But $4k > 2k+3$ if $k > 1$. Using these facts, it is not too hard to form an mdn of type $[k_1, \cdots, k_{j-1}, k'_j, \cdots, k'_n]$ where $k'_i = 4$. (We leave this to the reader.) This proves the second claim. Q.E.D.

Proof of Theorem 2. By Lemma 3, only the construction of the required mdn's remains and, in fact, only those which are of size between $4n$ and $4(n-1)$; for, the others are obtained inductively from lower dimensions. We may obtain these by combining $(n-1)$ dn's of type [4] with one of the type [3], and $(n-2)$ of types [4] with one of types [6] and [5], respectively. Again, we leave the details to the reader (e.g., use the construction devices employed above).

Remark 1. The term "dual necklace" is derived from the fact that the dual of the theorem of Sanders, concerning *shortest* rectilinear polygons, yields what intuitively corresponds to a necklace. Both theorems are in fact true, with the obvious modifications, in any geodesic metric space, including, of course, S^n (see **[1]** for proof).

Remark 2. The arguments given in Lemmas 1, 2, and 3 work equally well if instead of parallelopipeds in R^n we consider parallel, centrally symmetric, convex $2n$-gons in the plane. (These are the spheres of certain normed linear spaces.) Two of these will intersect if and only if the connected regions formed by extending both pairs of corresponding parallel sides intersect. Therefore, we need only set up a pencil of axes in the plane, that is, lines perpendicular to the sides of the $2n$-gons. Then, in the relevant proofs, use the intersections of the strip-like regions with the corresponding perpendicular axis. Thus, *sets of k parallel, centrally symmetric* $2n$*-gons which satisfy* (1) *and* (2) *do not exist if* $k>2n+3$; *those satisfying condition* (1) *do not exist if* $k>4n$.

REMARK 3. We close with the following question (raised by the referee): Given a dual or multi-dual necklace of k subsets, k satisfying certain bounds, under what conditions is it a necklace of *some* metric space spheres?

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