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PSEUDO-ISOTOPIES OF ARCS AND KNOTS

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ABSTRACT. The purpose of this paper is to show that if an arc or simple closed curve contains a nontrivial Wilder arc, then it is not possible to transform a straight line segment onto the arc or a knot onto the simple closed curve. The proof uses the fact that every knot has a unique finite decomposition into prime knots.

A *pseudo-isotopy* is a homotopy F_t, $0 \leqq t \leqq 1$, such that F_t is a homeomorphism for $t<1$. An arc A is *locally unknotted* [4] if for each $x \in A$ there is a neighborhood of x in A that lies on a disk. It is known [5], [6], [7] that if M is a 2-manifold, a locally unknotted arc or a locally unknotted simple closed curve in E^3 and if $\varepsilon>0$, then there is a polyhedral manifold $N \subset E^3$, homeomorphic to M, and an ε-pseudo-isotopy F_t of E^3 that transforms N onto M. In addition, $F_0=1$, $F_1|N$ is a homeomorphism of N onto M and the set of points whose preimages under F_1 are nondegenerate is a zero-dimensional subset of the set of wild points of M.

An arc is *mildly wild* if it is the union of two tame arcs. A *Wilder arc* [3] is a mildly wild locally peripherally unknotted [4] arc. See Example 1.4 of [1]. The following theorems show that the result above cannot be extended to all arcs and simple closed curves.

THEOREM 1. *If A is an arc that contains a nontrivial Wilder arc, I is a straight line segment, and $F_t: E^3 \to E^3$ is a pseudo-isotopy, then $F_1(I) \neq A$.*

THEOREM 2. *If S is a simple closed curve that contains a nontrivial Wilder arc, k is a knot, and $F_t: E^3 \to E^3$ is a pseudo-isotopy, then $F_1(k) \neq S$.*

However, it is trivial that any arc or simple closed curve can be transformed by a pseudo-isotopy onto a straight line segment or a circle, respectively.

Theorem 1 follows immediately from Theorem 2.

PROOF OF THEOREM 2. Let C be a geometric cone in E^3 with vertex p and square base F_0. For each integer $i>0$, let F_i be the intersection of C and the plane parallel to F_0 that is half way between F_{i-1} and p. Let C_i be

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the closed subset of C between F_i and F_{i+1}. Since S contains a nontrivial Wilder arc, we can assume without loss of generality that $S \cap C_0$ is a straight line segment A_0 and, for $i>0$, that $S \cap C_i$ is a polyhedral arc A_i that meets the boundary of C_i only at the endpoints of A_i and the mid-points of F_i and F_{i+1} such that A_i plus an arc on the boundary of C_i is a nontrivial knot.

Suppose there is a knot k in E^3 and a pseudo-isotopy $F_t: E^3 \to E^3$ such that $F_1(k)=S$. For each integer $i \geqq 0$, let N_i be a regular neighborhood of A_i in C_i such that $N_i \cap F_{i+1} = N_{i+1} \cap F_{i+1}$.

Let n be a positive integer. There exists a $t<1$ such that

$$F_t(k) \cap \left(\bigcup_0^{n+1} C_i \right) \subset \bigcup_0^{n+1} N_i.$$

We may suppose without loss of generality that $F_t(k) \cap C_0 = A_0$. Then there is a homeomorphism $h: E^3 \to E^3$ such that

$$h(A_0) = \bigcup_0^n A_i = h(F_t(k)) \cap \left(\bigcup_0^n C_i \right).$$

Since k and $hF_t(k)$ are equivalent, k can be decomposed into at least n knots for every n. But every knot has a unique finite decomposition into prime knots [**2**]. Hence no such pseudo-isotopy exists.

References

1. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990. MR **10**, 317.

2. R. H. Fox, *A quick trip through knot theory*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 120–167. MR **25** #3522.

3. R. H. Fox and O. G. Harrold, *The Wilder arcs*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 184–187. MR **25** #3519.

4. O. G. Harrold, Jr., *Combinatorial structures, local unknottedness, and local peripheral unknottedness*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 71–83. MR **28** #1597.

5. L. V. Keldyš, *Topological imbeddings and pseudo-isotopy*, Dokl. Akad. Nauk SSSR **169** (1966), 1262–1265=Soviet Math. Dokl. **7** (1966), 1078–1082. MR **34** #3556.

6. ———, *Topological imbeddings in a manifold and pseudoisotopy*, Mat. Sb. **71** (**113**) (1966), 433–453; English transl., Amer. Math. Soc. Transl. (2) **84** (1969), 253–275. MR **34** #6784.

7. ———, *Topological imbeddings of simple arcs and closed curves in E^3*, Dokl. Akad. Nauk SSSR **185** (1969), 513–516=Soviet Math. Dokl. **10** (1969), 376–379. MR **39** #2141.

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