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A PROPERTY OF ARITHMETIC SETS

HISAO TANAKA

ABSTRACT. We shall show that every nonempty countable arithmetic subset of N^N contains at least one element α such that the singleton $\{\alpha\}$ itself is arithmetic. The proof is carried out by using a method in classical descriptive set theory.

It is known that (*) if no member of a nonempty Σ_1^1 set E is hyperarithmetic then E contains a perfect subset. (In this note, sets mean subsets of N^N—the set of all 1-place number-theoretic functions which we identify with Baire zero-space.) In fact, every Σ_1^1 set with a nonhyperarithmetic element contains a perfect subset. (See, e.g., Harrison [**2**, Theorem 2.12] and Mathias [**4**, T3200].) In what follows, we shall show that an arithmetic counterpart of the proposition (*) holds true:

THEOREM 1. *If no member of a nonempty arithmetic set A is an arithmetic singleton, then A contains a perfect subset.*

It is evident that one can not replace "arithmetic singleton" by "arithmetic element" in our theorem.

T. G. McLaughlin has asked the following question (unpublished): Let A be a nonempty countable arithmetic set. Then, must some member of A be an arithmetic singleton? Now we can obtain an affirmative answer to this question as a direct corollary of our theorem, thus:

COROLLARY 2. *If A is a nonempty countable arithmetic set, then A contains at least one arithmetic singleton.*

Since every uncountable arithmetic set (in fact, every classical uncountable analytic set) contains a perfect subset, Corollary 2 is equivalent to Theorem 1. I do not know whether every member of a countable arithmetic set is an arithmetic singleton. This is also a problem presented by McLaughlin.

PROOF OF THEOREM 1. We shall illustrate for the case that A is a Π_5^0 set. Proof is analogous for the other cases. Note that if A is a Σ_{n+1}^0 set then we can reduce it to the case of Π_n^0.

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Now let A be a set defined by

$$A = \{\alpha \in N^N \mid (\forall x_0)(\exists y_0)(\forall x_1)(\exists y_1)R(\alpha, x_0, x_1, y_0, y_1)\},$$

where R is Π^0_1. Then we have

$$\begin{aligned}\alpha \in A &\Leftrightarrow (\exists\beta_0)(\exists\beta_1)(\forall x_0)(\forall x_1)R(\alpha, x_0, x_1, \beta_0(x_0), \beta_1(x_0, x_1))\\ &\Leftrightarrow (\exists\beta)(\forall x)R(\alpha, (x)_0, (x)_1, \beta(\langle(x)_0\rangle), \beta(\langle(x)_0, (x)_1, 1\rangle)),\end{aligned}$$

where $\langle a_0, a_1, \cdots, a_k\rangle = p_0^{a_0}\cdot p_1^{a_1}\cdots p_k^{a_k}$ and p_i is the $(i+1)$st prime number. (For notations used in this note, we mostly borrow from Kleene [3].) Let R' and R'' be predicates defined as follows:

$$\begin{aligned}R'(\alpha, s) \Leftrightarrow [&\mathrm{Seq}(s) \wedge \mathrm{Lh}(s) = \langle(\mathrm{Lh}(s))_0, (\mathrm{Lh}(s))_1, 2\rangle\\ &\rightarrow R(\alpha, (\mathrm{Lh}(s))_0, (\mathrm{Lh}(s))_1, \exp(s, \langle(\mathrm{Lh}(s))_0\rangle) - 1,\\ &\qquad \exp(s, \langle(\mathrm{Lh}(s))_0, (\mathrm{Lh}(s))_1, 1\rangle) - 1),\end{aligned}$$

where $\exp(s, i) = (s)_i$. And

$$R''(\alpha, s) \Leftrightarrow (\forall i)_{i \leq \mathrm{Lh}(s)} R'(\alpha, \mathrm{rstr}(s, i)),$$

where

$$\begin{aligned}\mathrm{rstr}(s, i) &= \prod_{k<i} p_k^{(s)_k}, \quad \text{if } \mathrm{Seq}(s) \wedge i \leqq \mathrm{Lh}(s),\\ &= 1, \qquad\qquad \text{otherwise.}\end{aligned}$$

Then R'' has the following properties:

(1) $\alpha \in A \Leftrightarrow (\exists\beta)(\forall x)R''(\alpha, \bar\beta(x))$,

(2) R'' is Π^0_1 and hence for each sequence number s, the set $E_s = \{\alpha \mid R''(\alpha, s)\}$ is a *closed* set, and

(3) the Souslin system $\mathfrak{S} = \{E_s \mid \mathrm{Seq}(s)\}$ is monotonic; that is, for all β and x, $E_{\bar\beta(x+1)} \subseteq E_{\bar\beta(x)}$.

Now, as is usual with classical descriptive set theory, for a given sequence number $\bar\gamma(m)$, we shall define a set $A^{\bar\gamma(m)}$ as follows:

(4) $$\alpha \in A^{\bar\gamma(m)} \Leftrightarrow (\exists\beta)(\forall x)R''(\alpha, \bar\gamma(m) * \bar\beta(x)).$$

Then we have

$$\begin{aligned}\alpha \in A^{\bar\gamma(m)} &\Leftrightarrow (\exists\beta)(\forall x)(\forall i)_{i\leq m+x} R'(\alpha, \mathrm{rstr}(\bar\gamma(m) * \bar\beta(x), i))\\ &\Leftrightarrow (\forall i)_{i \leq m} R'(\alpha, \bar\gamma(i)) \wedge (\exists\beta)(\forall i)R'(\alpha, \bar\gamma(m) * \bar\beta(i))\\ &\Leftrightarrow (\forall i)_{i\leq m}[i = \langle(i)_0, (i)_1, 2\rangle\\ &\qquad \rightarrow R(\alpha, (i)_0, (i)_1, \gamma(\langle(i)_0\rangle), \gamma(\langle(i)_0, (i)_1, 1\rangle))]\\ &\quad \wedge (\exists\beta)(\forall i)[m + i = \langle(m+i)_0, (m+i)_1, 2\rangle\\ &\qquad \rightarrow. \{\langle(m+i)_0, (m+i)_1, 1\rangle < m\\ &\qquad\quad \rightarrow R(\alpha, (m+i)_0, (m+i)_1, \gamma(\langle(m+i)_0\rangle),\\ &\qquad\qquad \gamma(\langle(m+i)_0, (m+i)_1, 1\rangle))\}\end{aligned}$$

$$\begin{aligned}
&\wedge \{\langle (m+i)_0\rangle < m \wedge \langle (m+i)_0, (m+i)_1, 1\rangle \geqq m \\
&\quad \rightarrow R(\alpha, (m+i)_0, (m+i)_1, \gamma(\langle (m+i)_0\rangle), \\
&\qquad\qquad \beta(\langle (m+i)_0, (m+i)_1, 1\rangle - m))\} \\
&\wedge \{\langle (m+i)_0\rangle \geqq m \\
&\quad \rightarrow R(\alpha, (m+i)_0, (m+i)_1, \beta(\langle (m+i)_0\rangle - m), \\
&\qquad\qquad \beta(\langle (m+i)_0, (m+i)_1, 1\rangle - m))\}].
\end{aligned}$$

The second member of the outermost conjunction in the latter formula is equivalent to

$$\begin{aligned}
&(\exists\beta_0)(\exists\beta_1)(\forall x_0)(\forall x_1)[\{\langle x_0, x_1, 1\rangle < m \rightarrow R(\alpha, x_0, x_1, \gamma(\langle x_0\rangle), \gamma(\langle x_0, x_1, 1\rangle))\} \\
&\qquad \wedge \{\langle x_0\rangle < m \wedge \langle x_0, x_1, 1\rangle \geqq m \\
&\qquad\quad \rightarrow R(\alpha, x_0, x_1, \gamma(\langle x_0\rangle), \beta_1(\langle x_0, x_1, 1\rangle - m))\} \\
&\qquad \wedge \{\langle x_0\rangle \geqq m \\
&\qquad\quad \rightarrow R(\alpha, x_0, x_1, \beta_0, (\langle x_0\rangle - m), \beta_1(\langle x_0, x_1, 1\rangle - m))\}] \\
&\Leftrightarrow (\forall x_0)(\exists y_0)(\forall x_1)(\exists y_1)[\{\langle x_0, x_1, 1\rangle < m \\
&\qquad\quad \rightarrow R(\alpha, x_0, x_1, \gamma(\langle x_0\rangle), \gamma(\langle x_0, x_1, 1\rangle))\} \\
&\qquad \wedge \{\langle x_0\rangle < m \wedge \langle x_0, x_1, 1\rangle \geqq m \rightarrow R(\alpha, x_0, x_1, \gamma(\langle x_0\rangle), y_1)\} \\
&\qquad\qquad \wedge \{\langle x_0\rangle \geqq m \rightarrow R(\alpha, x_0, x_1, y_0, y_1)\}].
\end{aligned}$$

(Note that $\bar{\gamma}(m)$ is a given fixed sequence number.) Therefore, for each sequence number s, A^s is an arithmetic subset of N^N, too. Further, by the definition (4) we have

$$A^{[a_0, a_1, \cdots, a_k]} = \bigcup_{n=0}^{\infty} A^{[a_0, a_1, \cdots, a_k, n]}, \tag{5}$$

where we denote $\langle a_0+1, a_1+1, \cdots, a_k+1\rangle$ by $[a_0, a_1, \cdots, a_k]$.

Now suppose that no member of A constitutes an arithmetic singleton. Let $\alpha \in A$. Since $A = \bigcup_{n=0}^{\infty} A^{[n]}$, there is an n_0 such that $\alpha \in A^{[n_0]}$. $A^{[n_0]}$ does not contain any arithmetic singleton, since its overset A does not. As seen above, $A^{[n_0]}$ is also an arithmetic set and hence it contains no isolated elements. Therefore $A^{[n_0]}$ is dense-in-itself. So, for each number m_0, $A^{[n_0]} \cap \delta(\bar{\alpha}([m_0]))$ is nonempty and dense-in-itself, where $\delta(s)$ denotes the Baire interval determined by a sequence number s. Let us put

$$B^{[m_0]} = A^{[n_0]} \quad \text{and} \quad F_{[m_0]} = E_{[n_0]}$$

for all m_0. From each set $B^{[m_0]} \cap \delta(\bar{\alpha}([m_0]))$ we can choose an element $\alpha_{[m_0]}$ such that the $\alpha_{[m_0]}$'s satisfy the following conditions:

$$\alpha_{[m_0]} \neq \alpha, \quad \alpha_{[m_0]} \neq \alpha_{[m_0']} \quad \text{if } m_0 \neq m_0'.$$

Since $B^{[m_0]}=\bigcup_{n=0}^{\infty} A^{[n_0,n]}$, for each m_0 there is an n_1 such that $\alpha_{[m_0]}\in A^{[n_0,n_1]}$. Let us put

$$B^{[m_0,m_1]}=A^{[n_0,n_1]} \quad \text{and} \quad F_{[m_0,m_1]}=E_{[n_0,n_1]}$$

for all m_1. Then $B^{[m_0,m_1]}\cap\delta(\bar{\alpha}_{[m_0]}([m_0+m_1+1]))$ is nonempty and dense-in-itself. From each set $B^{[m_0,m_1]}\cap\delta(\bar{\alpha}_{[m_0]}([m_0+m_1+1]))$, we can choose an $\alpha_{[m_0,m_1]}$ such that $\alpha_{[m_0,m_1]}$'s satisfy the following conditions:

$$\alpha_{[m_0,m_1]}\neq\alpha; \qquad \alpha_{[m_0,m_1]}\neq\alpha_{[m_0']};$$
$$\alpha_{[m_0,m_1]}\neq\alpha_{[m_0',m_1']} \quad \text{if } [m_0,m_1]\neq[m_0',m_1'].$$

And so on. Thus we obtain elements $\alpha_{[m_0,m_1,\cdots,m_k]}\in A$ for k, $m_i=0,1,2,\cdots$, and they possess the following properties:

$$\text{(6)} \qquad \alpha_{[m_0,\cdots,m_k]}\neq\alpha_{[m_0',\cdots,m_j']} \quad \text{if } [m_0,\cdots,m_k]\neq[m_0',\cdots,m_j'],$$

$$\text{(7)} \qquad \alpha_{[m_0,\cdots,m_k]}\in B^{[m_0,\cdots,m_k]}=A^{[n_0,\cdots,n_k]}\subseteq E_{[n_0,\cdots,n_k]}=F_{[m_0,\cdots,m_k]},$$

where $[n_0,\cdots,n_k]$ is determined by $[m_0,\cdots,m_k]$,

$$\text{(8)} \qquad \alpha_{[m_0,\cdots,m_k,m_{k+1}]}\in\delta(\bar{\alpha}_{[m_0,\cdots,m_k]}([m_0+\cdots+m_{k+1}+k+1])).$$

Let $Q=\{\alpha_s|\mathrm{Seq}(s) \text{ and } \mathrm{Lh}(s)>0\}$. Then Q is dense-in-itself and hence its derived set Q' is a perfect set. Using (1)–(3) and (6)–(8) we can show that Q' is contained in A. In proving this fact, note that each E_s is a closed set. (For details, see Hahn [**1**, pp. 356–358].) Therefore A contains a perfect subset. This completes the proof of Theorem 1.

Since the final expression for $\alpha\in A^{\bar{y}(m)}$ in the preceding proof is also Π_5^0, we have shown that if A is a nonempty Π_5^0 set with no Π_5^0 singleton then A contains a perfect subset.[1] Thus we obtain the following theorem:

THEOREM 3. *Every nonempty countable Σ_{n+1}^0 set contains a Π_n^0 singleton.*

References

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College of Engineering, Hosei University, Tokyo, Japan

Department of Mathematics, University of Illinos, Urbana, Illinois 61801

[1] This is based on a suggestion of Professor McLaughlin.