PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

DIVINSKY'S RADICAL

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ABSTRACT. Let F and R be rings, M an F-R-bimodule, and Δ the largest F-submodule N of M such that for each $x \in N, fx = x$ for some $f \in F$.

(1) If either F or M satisfies the minimum condition then $\Delta = F^k M$ for some positive integer k; provided that whenever $x \in F^\omega M = \bigcap_{n=1}^{\infty} (F^n M)$ and $Fx \subseteq \Delta$, then $x \in \Delta$.

(2) If M satisfies the maximum condition and $F = (f_1, \cdots, f_n)$ where $f_1, \cdots, f_n$ is a *normalising* set of generators (that is,

$$f_i F = F f_i \text{ modulo } (f_1, \cdots, f_{i-1})$$

for each $i = 1, \cdots, n$), then $\Delta = F^\omega M$.

(3) If $M = F = R$, $\Delta = (0)$, R satisfies the maximum condition, and R has a normalising set of generators, then R can be embedded in a Jacobson radical ring.

1. **Introduction.** Let R be an associative ring, M a right R-module and F a ring of R-endomorphisms of M. We shall think of M as an F-R-bimodule and write both fx and $f(x)$ ($f \in F$ and $x \in M$), whichever is most convenient. There is a largest F-submodule N of M such that for each $n \in N, f(n) = n$ for some $f \in F$. This submodule is also an R-submodule and is denoted by $\Delta(F, M)$ [1] or simply Δ when no confusion can result. The submodule $\Delta(F, M)$ is the *Divinsky radical* of the pair (F, M). In [2], where Bostock and Patterson introduce this generalization of left D-regularity, it is shown that the submodule $\Delta(F, M)$ satisfies the basic properties of radicals.

Throughout this paper F, M, and R will retain these meanings. The two-sided ideal of F generated by elements $f_1, \cdots, f_n \in F$ will be denoted by $(f_1, \cdots, f_n)$. And we shall write min-l and max-l for "minimum condition on left ideals" and "maximum condition on left ideals" respectively.

EXAMPLES. (i) Let B and R be any two rings and M a B-R-bimodule. Set F=the ring of R-endomorphisms of M which are determined by

Received by the editors February 17, 1971.

AMS 1970 *subject classifications*. Primary 16-00, 16A21; Secondary 16A22, 16A46, 16A64.

Key words and phrases. Divinsky's radical, D-regularity, embed in a Jacobson radical ring, intersection theorem, AR property, quasi quotient ring.

[1] Since M is a left F-module this notation seems more natural than $\Delta(M, F)$ which is used in [2].

elements of B; F is the ring of all $l_b: x \to bx$ where $b \in B$. The Divinsky radical $\Delta(F, M)$ is the largest B-submodule N of M such that for each $x \in N$, $bx = x$ for some $b \in B$. Since no confusion can result we shall always write $\Delta(B, M)$ in this situation.

(ii) An important case is when $M = B = R$ and the action of the rings on the module is just ring multiplication. Then $\Delta(B, M)$ $(= \Delta(R, R))$ is a two-sided ideal of R. This is the original situation which was studied by Divinsky [**3**].

Notice that we do not obtain a radical in the sense of [**4**]. This is because $\Delta(\Delta(R, R), \Delta(R, R))$ may not equal $\Delta(R, R)$; see [**3**, Example 1].

2. The minimum condition.

PROPOSITION 2.1 (BOSTOCK AND PATTERSON [**2**]). *Assume that F has min-l. Let $e \in F$ be an idempotent such that $f - fe$ is nilpotent for all $f \in F$. Then $ex = x$ for all $x \in \Delta$.*

Recall that $\Delta = \Delta(F, M)$ is the largest F-R-submodule of M such that for each $x \in \Delta$, $fx = x$ for some $f \in F$.

THEOREM 2.2. *Assume that F has min-l and a central idempotent e such that $e + N$ is the identity of the factor ring F/N, where N is the nil radical of F, then*:

(i) $\Delta = eM$,

(ii) $M = \Delta \oplus A$ *where* $A = \{x \in M : ex = 0\}$,

(iii) $\Delta = F^k M$ *and* $F^k A = (0)$ *for some positive integer* k.

PROOF. From Proposition 2.1 we see that $ex = x$ for all $x \in \Delta$, so $\Delta \subseteq eM$. Since e is central, eM is an F-submodule; thus $eM \subseteq \Delta$. Therefore $\Delta = eM$.

Let $A = \{x \in M : ex = 0\}$. Since e is central, A is an F-submodule of M. It is straightforward to check that $M = \Delta \oplus A$.

Choose a positive integer k such that $N^k = (0)$. Now $F = Fe + N$ and e is central so

$$Fe \subseteq F^k = (Fe + N)^k \subseteq Fe + N^k = Fe.$$

Thus $F^k = Fe$ so it is immediate that $\Delta = F^k M$ and $F^k A = (0)$. This completes the proof.

Theorem 2.2 presents "ideal" or "model" results. It is interesting to compare the following theorems and to consider how the relatively weaker hypotheses lead to relatively poorer approximations to these results.

In the next theorem we shall be dealing with rings F which may not have central idempotents. However, we shall impose the following condition on the pair (F, M):

$$(*) \qquad F^\omega M \cap \{a \in M : Fa \subseteq \Delta\} \subseteq \Delta$$

where $F^{\omega}M=\bigcap_{n=1}^{\infty}(F^nM)$. Notice that $\Delta\subseteq F^{\omega}M$ and $\Delta\subseteq\{a\in M: Fa\subseteq\Delta\}$. Clearly, condition (*) is satisfied if $\Delta=F^{\omega}M$; so, for rings F which satisfy the conditions of Theorem 2.2, every pair (F, M) satisfies condition (*).

The converse is not true. To see this let F be any ring with a left identity e but with no central idempotents. For any left F-module M, $FM\subseteq\Delta$ since $e(fx)=(ef)x=fx$ for all $f\in F$ and $x\in M$. Clearly $\Delta=F\Delta\subseteq FM$ so $\Delta=FM$. Thus (F, M) satisfies condition (*). Now F is a ring such that every pair (F, M) satisfies condition (*), but F has no central idempotents. Of course there are such rings F which have min-l. The simplest examples are obtained by considering the algebra (over any field) with basal elements e and b where $e^2=e$, $eb=b$, $b^2=0$, and $be=0$.

THEOREM 2.3. *Assume that F has min-l. If the pair (F, M) satisfies condition* (*), *then $\Delta=F^kM$ for some positive integer k.*

PROOF. Suppose that $F^kM=F^{\omega}M\supsetneqq\Delta$. Choose $x\in F^kM$ such that $x\notin\Delta$. The pair (F, M) satisfies condition (*) so $Fx\nsubseteq\Delta$. Choose L minimal among all left ideals I of F for which $Ix\nsubseteq\Delta$.

Let $l\in L$. If $lx\notin\Delta$ then, by condition (*), $Flx\nsubseteq\Delta$. So by the minimality of L, $Fl=L$. It follows that for each $l\in L$ there is an $f\in F$ such that $f(lx)=lx$. Thus $Lx\subseteq\Delta$. This is a contradiction; the theorem follows.

Not all rings F with min-l are such that all pairs (F, M) satisfy condition (*); even when $M=F=R$. To see this let F be the algebra (over any field) with basal elements e and b where $e^2=e$, $eb=0$, $b^2=0$ and $be=b$. Notice that F has only three (two-sided) ideals: $(0)\subseteq(b)\subseteq(b, e)=F$. Now one observes that $F^{\omega}=F$, $\Delta=(0)$ and $b\in\{a\in F: Fa\subseteq\Delta\}$.

When $M=F=R$, Theorem 2.3 provides another proof of Theorem 5 in **[3]**.

THEOREM 2.4 *Assume that M satisfies the minimum condition on F-submodules. If the pair (F, M) satisfies condition* (*), *then $\Delta=F^kM$ for some positive integer k.*

The proof is very similar to that of Theorem 2.3 and will not be given.

In Theorem 2.3 we are dealing with rings F with min-l, so by Proposition 2.1 there is an element $e\in F$ which leaves Δ fixed. The following example shows that this need not be the case when we assume only that M satisfies the minimum condition on F-submodules.

EXAMPLE 2.5. Let p and q be distinct prime numbers. Set $F=R=$the ideal of the ring of integers which is generated by q. Put $M=p^{\infty}$ and make M an F-R-bimodule in the obvious way (p^{∞} is the group $Z[1/p]/Z$ with trivial multiplication; see **[4**, p. 14]). Now $\Delta=p^{\infty}$ but no element of F leaves p^{∞} fixed.

There are pairs (F, M) such that M satisfies the minimum condition on F-submodules, and which do not satisfy condition (*). The example immediately following Theorem 2.3 illustrates this. A second example can be obtained by considering Example 2.5 modified by setting $p=q$. Then $F^\omega M=M$, $\Delta=(0)$ and $1/p\in\{a\in M:Fa\subseteq\Delta\}$.

3. **The maximum condition.** In this section we shall consider some situations in which $\Delta=F^\omega M$. Our results and methods of proof follow those which are concerned with the Krull Intersection Theorem for non-commutative rings; see Nouazé and Gabriel [**8**, 2.7], McConnell [**6**], [**7**], and Smith [**9**].

A ring F has the *AR property* if and only if for each left ideal E of F, $F^n\cap E\subseteq FE$ for some positive integer n. The pair (F, M) has the *AR property* if and only if for each F-submodule N of M, $F^nM\cap N\subseteq FN$ for some positive integer n.

The proof of the following proposition is standard—see, for example, [**7**, Lemma 4].

PROPOSITION 3.1. *If (F, M) has the AR property then $\Delta=F^\omega M$.*

McConnell [**7**] gives the following definition. Let $F=(f_1,\cdots,f_n)$. Then $f_1,\cdots,f_n$ is a *normalising set of generators of F* if and only if:

(i) $f_1F=Ff_1$,

(ii) $f_iF=Ff_i$ modulo $(f_1,\cdots,f_{i-1})$ for each $i=2,\cdots,n$.

THEOREM 3.2. *If F has a normalising set of generators and M satisfies the maximum condition on F-submodules, then the pair (F, M) has the AR property.*

PROOF (SEE [**7**, LEMMA 8], [**8**, 2.7]). Let N be an F-submodule of M. Now $FN\subseteq N$ so $FN\cap N=FN$. Choose an F-submodule $M'\supseteq FN$ such that $M'\cap N=FN$ and M' is maximal with this property. One easily checks that since M' is maximal, N/FN is an essential submodule of M/M'. Recall that a submodule is *essential* if it has a nonzero intersection with every nonzero submodule. Now if we can find a positive integer l such that $F^l(M/M')=(0)$, then $F^lM\cap N\subseteq FN$ as is required.

Thus it is sufficient to show that if N is an essential F-submodule of M and $F^sN=(0)$ for some positive integer s (in our case $s=1$), then $F^lM=(0)$ for some positive integer l.

Let $f_1,\cdots,f_n$ be a normalising set of generators of F and for each $i\geqq 1$, define $K_i=\{x\in M:f_1^ix=0\}$. Since $f_1F=Ff_1$, $f_1^iF=Ff_1^i$ for each integer $i\geqq 1$. Thus each K_i is an F-submodule. Since M satisfies the maximum condition we may choose $h\geqq s$ so that $K_i=K_h$ for all $i\geqq h$.

Suppose that $y=f_1^h x \in f_1^h M \cap K_h$. Then $f_1^h y = f_1^{2h} x = 0$ so $x \in K_{2h} = K_h$. Thus $y=0$, so $f_1^h M \cap K_h = (0)$. The set $f_1^h M$ is an F-submodule because $Ff_1^h = f_1^h F$. And since $h \geqq s$, $K_h \supseteq N$. Now, N is essential so $f_1^h M = (0)$. Thus we have a finite chain of F-submodules:

$$(0) \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_h = M.$$

We proceed by induction on n. If $n=1$, then $F^h = (f_1)^h = Ff_1^h$. Therefore, $F^h M = (0)$.

Let $n>1$ and assume that the theorem is true for any ring with a normalising set of generators of fewer than n elements. Since $f_1 K_1 = (0)$ we may consider K_1 an $F/(f_1)$-module. Clearly $K_1 \cap N$ is an essential submodule of K_1 so by our induction hypothesis there is an integer p such that $F^p K_1 = (0)$. In fact, $F^p K_{i+1} \subseteq K_i$ for all $i \geqq 1$. To see this simply notice that

$$f_1^i(F^p K_{i+1}) = (f_1^i F^p) K_{i+1} = (F^p f_1^i) K_{i+1} = F^p(f_1^i K_{i+1})$$

and that $f_1^i K_{i+1} \subseteq K_1$. Therefore, $F^{hp} M = (0)$. The proof is complete.

COROLLARY 3.3. *If F has a normalising set of generators and M satisfies the maximum condition on F-submodules, then* $\Delta = F^\omega M$.

In the case that $M=F=R$ the theorem concludes that for each left ideal E of F, $F^{n+1} \cap E \subseteq FE$ for some positive integer n. Thus the ring F has the AR property.

For $M=F=R$ the corollary reads: *Let F be a ring with max-l and a normalising set of generators. Then* $\Delta = F^\omega$.

4. **Radical quasi quotient rings.** Let R be a ring with min-l such that R^ω contains no nonzero right annihilators of R. Theorem 2.3 states that if $\Delta=(0)$, then R is nilpotent. However, there are rings with both max-l and min-l and with $\Delta=(0)$, but which contain idempotents (see the example immediately following Theorem 2.3). Not only are these rings not nilpotent, but they cannot be embedded in Jacobson radical rings. We have just seen that $\Delta = R^\omega$ if R has max-l and a normalising set of generators. In this section we shall prove that if R has max-l, a normalising set of generators, and $\Delta=(0)$, then R can be embedded in a Jacobson radical ring.

Let R be a ring and $x, y \in R$. Define $x \circ y = x + y - xy$. When $x \circ y = 0 = y \circ x$ we say that x is *radical* and y is the *quasi-inverse* of x. The element x is *semiradical* if for $w \in R$, $xw = w$ implies $w=0$ and $wx = w$ implies $w=0$. The ring R is *radical* (*semiradical*) if x is radical (semiradical) for all $x \in R$. A ring $Q \supseteq R$ is a *left quasi quotient ring* of R if the following conditions are satisfied:

(i) If $a \in R$ and a is semiradical then a is radical in Q.

(ii) Every element $x \in Q$ is of the form $a' \circ b$ where $a, b \in R$, a is semiradical, and a' is the quasi-inverse of a.

THEOREM A (ANDRUNAKIEVIČ [1]). *A ring R possesses a left quasi quotient ring Q which is a radical ring if and only if R is semiradical and for $a, b \in R$ there exists $a_1, b_1 \in R$ such that $b_1 \circ a = a_1 \circ b$.*

Kroener [5] gives the above theorem and several other conditions which are equivalent to the existence of a radical left quasi quotient ring. But these conditions all involve a reference to '$\circ$', quasi multiplication, and are thus difficult to verify. The following theorem of Smith allows us to find a relatively weak sufficient condition, which does not involve a reference to quasi multiplication, for the existence of a radical left quasi quotient ring.

THEOREM S (SMITH [9]). *Let R be a ring with max-l and an identity element 1. Suppose that R has an ideal I which, as a ring, has the AR property. Then given $r \in R$ and $i \in I$ there exists $r' \in R$ and $i' \in I$ such that $r'(1-i)=(1-i')r$.*

THEOREM 4.1. *Let R be a ring with max-l. If R has the AR property and $\Delta=(0)$ then R has a radical left quasi quotient ring.*

PROOF. Embed R into a ring R^1 with identity element 1 in the usual way. Suppose $a, b \in R$. Now R^1 has max-l so we may apply Theorem S with $r=1-b$ and $i=a$. Thus we obtain $r' \in R^1$ and $i' \in R$ such that $r'(1-a)=(1-i')(1-b)$. Set $a_1=i'$ and $b_1=b+i'-i'b-r'a$. Then $(1-b_1)(1-a)=(1-a_1)(1-b)$ so $b_1 \circ a = a_1 \circ b$.

By Proposition 3.1, $\Delta = R^\omega$ so $R^\omega=(0)$. This implies that R is semiradical because if $xw=w$ or $wx=w$ then $w \in R^\omega$.

Both conditions of Theorem A are satisfied so R has a radical left quasi quotient ring.

COROLLARY 4.2. *Let R be a ring with max-l and a normalising set of generators. If $\Delta=(0)$, then $R^\omega=(0)$ and R has a radical left quasi quotient ring.*

Let R be a subring of a radical ring S. If $x \in \Delta = \Delta(R, R)$, then there is an $r \in R$ such that $rx=x$. Since S is a radical ring there is a $y \in S$ such that $r \circ y = y \circ r = 0$. Multiplying on the right by x we see that:

$$0 = yx + rx - yrx = yx + x - yx = x.$$

Therefore, $\Delta=(0)$.

COROLLARY 4.3. *Let R be a ring with max-l and a normalising set of generators. Then $R^\omega=(0)$ if and only if R can be embedded in a radical ring.*

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