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AN EXTENSION OF THE NOETHER-DEURING THEOREM

KLAUS W. ROGGENKAMP

Abstract. Let R be a commutative semilocal noetherian ring, Λ a left noetherian R-algebra and M, N finitely generated left Λ-modules such that $\mathrm{End}_\Lambda(M)$ is of finite type over R. By $\hat{R}$ we denote the (rad R)-adic completion of R.

Theorem. *M is Λ-isomorphic to a direct summand of N iff $\hat{R}\otimes_R M$ is $\hat{R}\otimes_R\Lambda$-isomorphic to a direct summand of $\hat{R}\otimes_R N$.*

This result is used to prove a generalization of the

Noether-Deuring theorem. *Let S be a commutative R-algebra which is a faithful projective R-module of finite type; then M is Λ-isomorphic to direct summand of N iff $S\otimes_R M$ is $S\otimes_R \Lambda$-isomorphic to a direct summand of $S\otimes_R N$.*

Let R be a semilocal commutative noetherian ring with Jacobson radical $J(R)$ and denote by $\hat{R}$ the $J(R)$-adic completion of R; let S be a commutative R-algebra such that $\hat{S}=\hat{R}\otimes_R S$ is a faithful projective $\hat{R}$-module of finite type. As a generalization of the Noether-Deuring theorem for integral representations we shall prove

Theorem I. *Let Λ be a left noetherian R-algebra, and M, N finitely generated left Λ-modules such that* $\mathrm{End}_\Lambda(M)$ *is of finite type over R. Then M is Λ-isomorphic to a direct summand of N if and only if $S\otimes_R M$* is *$S\otimes_R\Lambda$-isomorphic to a direct summand of $S\otimes_R N$.*

It has been pointed out to me by the referee that this is part of a result of A. Grothendieck [**8**, Proposition 2.5.8.(a)]. A similar statement has also been proven by Białynicki-Birula [**4**] using noncommutative Amitsur cohomology and the "théorie de déscente". Our theorem here is the result of an attempt to give a simplified proof of one of the theorems in Białynicki-Birula's paper.

We shall keep the notation introduced above throughout the paper, and to simplify the notation, we shall write $X|Y$ to indicate that X is isomorphic to a direct summand of Y. The key role in our proof of

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Theorem I is played by

THEOREM II. *Let Λ be a left noetherian R-algebra, and M, N finitely generated left Λ-modules such that* $\mathrm{End}_\Lambda(M)$ *is of finite type over R. Then*

$$M \mid N \text{ as } \Lambda\text{-modules if and only if}$$
$$\hat{R} \otimes_R M \mid \hat{R} \otimes_R N \text{ as } \hat{R} \otimes_R \Lambda\text{-modules.}$$

We *remark* that in both theorems the condition that $\mathrm{End}_\Lambda(M)$ is of finite type over R is surely satisfied if M is of finite type over R; in fact, $R^{(s)} \to M \to 0$ exact, implies $0 \to \mathrm{End}_R(M) \to M^{(s)}$ exact, and so $\mathrm{End}_R(M)$ is of finite type over R, R being noetherian. But $\mathrm{End}_\Lambda(M) \hookrightarrow \mathrm{End}_R(M)$ and so $\mathrm{End}_\Lambda(M)$ is of finite type over R.

PROOF OF THEOREM II. It suffices to prove one direction. So let us assume $\hat{R} \otimes_R M \mid \hat{R} \otimes_R N$ as $\hat{R} \otimes_R \Lambda$-modules. This is equivalent to the existence of a split monomorphism

$$0 \longrightarrow \hat{R} \otimes_R M \xrightarrow{\hat{\sigma}} \hat{R} \otimes_R N.$$

Since $\hat{R}$ is a faithfully flat R-module (cf. Bourbaki [**6**, Chapitre III, §3, N° 5]) and since M is a finitely generated left module over the left noetherian ring Λ, we have natural isomorphisms (cf. Auslander-Goldman [**1**, Lemma 2.4]) for any left Λ-module X.

(1) $\hat{R} \otimes_R \mathrm{Ext}^i_\Lambda(M, X) \cong \mathrm{Ext}^i_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R M, \hat{R} \otimes_R X)$ for $1 = 0, 1, \cdots$.

Using this isomorphism for $i=0$ and identifying both structures, we may write

$$\hat{\sigma} = \sum_{i=1}^{n} \hat{r}_i \otimes \sigma_i, \qquad \hat{r}_i \in \hat{R},\ \sigma_i \in \mathrm{Hom}_\Lambda(M, N),\ 1 \leqq i \leqq n.$$

However, $R/J(R) \cong \hat{R}/\hat{R} \otimes_R J(R)$, and so we can find elements $r_i \in R$, $1 \leqq i \leqq n$, such that $1^{\wedge} \otimes r_i - \hat{r}_i \in \hat{R} \otimes_R J(R) = J(\hat{R})$. To prove Theorem II, we have to establish the existence of a split monomorphism $0 \to M \xrightarrow{\sigma} N$. We claim that $\sigma = \sum_{i=1}^n r_i \varphi_i \in \mathrm{Hom}_\Lambda(M, N)$ has the desired properties. Since $\hat{R}$ is a faithfully flat R-module, it suffices to show that $1_{\hat{R}} \otimes_R \sigma$ is a split monomorphism. In fact, assuming that $1_{\hat{R}} \otimes_R \sigma$ is a split monomorphism, σ must be monic, and it remains to show that the sequence

$$E : 0 \longrightarrow M \xrightarrow{\sigma} N \longrightarrow N/\mathrm{Im}\,\sigma \to 0$$

is split exact. We consider the R-submodule X of $\mathrm{Ext}^1_\Lambda(N/\mathrm{Im}\,\sigma, M)$ generated by the class of E. Because of the isomorphism (1) for $i=1$ we have $\hat{R} \otimes_R X = 0$ and so X must be zero; i.e., E is split exact. It remains to show that $1_{\hat{R}} \otimes \sigma$ is a split monomorphism. Since $\hat{\sigma}$ was a split monomorphism to start with, there exists $\hat{\tau} \in \mathrm{Hom}_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R N, \hat{R} \otimes_R M)$ such that $\hat{\sigma}\hat{\tau}=1_{\hat{R}\otimes_R M}$. But then

$$(1_{\hat{R}} \otimes \sigma)\hat{\tau} - 1_{\hat{R}\otimes_R M} = (1_{\hat{R}} \otimes \sigma - \hat{\sigma})\hat{\tau} = \left[\sum_{i=1}^{n}(r_i - \hat{r}_i) \otimes \sigma_i\right]\hat{\tau} \in J(\hat{R})\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{R} \otimes_R M).$$

But $\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes_R M)$ is of finite type over $\hat{R}$ and so $J(\hat{R})\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes M)$ is contained in the Jacobson radical of $\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes_R M)$ (cf. Bourbaki [**5**, Chapitre VIII, §6, N° 3, Théorème 2]) and so $(1_{\hat{R}}\otimes\sigma)\hat{\tau}$ is a unit in $\mathrm{End}_{\hat{R}\otimes\Lambda}(\hat{R}\otimes_R M)$; i.e., $1_{\hat{R}}\otimes\sigma$ is a split monomorphism and so $M|N$. Q.E.D.

COROLLARY 1. *Let Λ be an R-algebra and M a finitely presented left Λ-module such that* $\mathrm{End}_\Lambda(M)$ *is of finite type over R and N a left Λ-module. Then $M\cong_\Lambda N$ if and only if $\hat{R}\otimes_R M\cong\hat{R}\otimes_R N$.*

The *proof* is similar to the one of Theorem II; however, we do not need the assumption that Λ is left noetherian, since (1) is valid for $i=0$ also for a finitely presented Λ-module M.

REMARK. Under the assumptions of Corollary 1, the Krull-Schmidt theorem is valid for the indecomposable direct summands of $\hat{R}\otimes_R M$. For this it suffices to know, that for each indecomposable direct summand $\hat{X}$ of $\hat{R}\otimes_R M$, the ring $\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{X})$ is complete with respect to the topology induced by $J(\hat{R})\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{X})$ (cf. Bass [**3**, Chapter III, Proposition 2.10]); but this is clear since $\mathrm{End}_{\hat{R}\otimes_R\Lambda}(\hat{X})$ is of finite type over $\hat{R}$ (cf. Bourbaki [**6**, Chapitre III, §3, N° 4, Théorème 3]).

COROLLARY 2. *Under the assumptions of Corollary* 1, *let X be a finitely presented left Λ-module such that* $\mathrm{End}_\Lambda(M\oplus X)$ *is of finite type over R. Then $M\oplus X\cong N\oplus X$ implies $M\cong N$.*

PROOF. This is an immediate consequence of Corollary 1 and the Remark.

COROLLARY 3. *Under the assumptions of Corollary* 1, $M^{(n)}\cong N^{(n)}$ *implies $M\cong N$.*

PROOF. This follows from Corollary 1 and the Remark.

COROLLARY 4. *Let $\mu_\Lambda(X)$ denote the minimal number of Λ-generators of the left Λ-module X, where Λ is an R-algebra. Assume that M is a finitely presented left Λ-module which is of finite type over R. Then $\mu_\Lambda(M)\leqq n$ if and only if $\mu_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes_R M)\leqq n$.*

PROOF. Since the tensor product is right exact, it suffices to prove one direction. Let $\mu_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes_R M)\leqq n$. Then we have an epimorphism

$$(\hat{R} \otimes_R \Lambda)^{(n)} \xrightarrow{\hat{\sigma}} \hat{R} \otimes_R M \longrightarrow 0.$$

As in the proof of Theorem II, we construct $\sigma\in\mathrm{Hom}_\Lambda(\Lambda^{(n)}, M)$ such that

$$(1_{\hat{R}} \otimes \sigma) - \hat{\sigma} \in J(\hat{R})\mathrm{Hom}_{\hat{R}\otimes_R\Lambda}(\hat{R} \otimes_R \Lambda^{(n)}, \hat{R} \otimes_R M).$$

But then $\mathrm{Im}(1_{\hat{R}}\otimes\sigma)+J(\hat{R})(\hat{R}\otimes_R M)=\hat{R}\otimes_R M$ and Nakayama's Lemma shows that $1_{\hat{R}}\otimes\sigma$ must be an epimorphism. However, $\hat{R}\otimes_R$— is faithfully flat, and so φ is an epimorphism.

Finally we turn to the proof of Theorem I.

It suffices to prove one direction. Let $S\otimes_R M | S\otimes_R N$. Then $\hat{S}\otimes_R M|\hat{S}\otimes_R N$ as $\hat{S}\otimes_R\Lambda$-modules. However, $J(R)=\bigcap_{i=1}^s m_i$, where $\{m_i\}_{1\leqq i\leqq s}$ are the maximal ideals of R. Then $\hat{R}=\prod_{i=1}^s \hat{R}_i$ is the product of complete local rings. Since we have assumed $\hat{S}$ to be a faithful projective $\hat{R}$-module of finite type, we have

$$\hat{S} \simeq_{\hat{R}} \bigoplus_{i=1}^{s} \hat{R}_i^{(n)_i}, \qquad n_i > 0, \qquad 1 \leqq i \leqq s.$$

Thus $\hat{S}\otimes_R M|\hat{S}\otimes_R N$ as $\hat{S}\otimes_R\Lambda$-modules implies

$$(\hat{R}_i \otimes_R M)^{(n_i)} \mid (\hat{R}_i \otimes_R N)^{(n_i)}, \qquad 1 \leqq i \leqq s$$

as $\hat{R}_i\otimes_R\Lambda$-modules. Now the Krull-Schmidt theorem shows

$$\hat{R}_i \otimes_R M \mid \hat{R}_i \otimes_R N, \qquad 1 \leqq i \leqq s,$$

and so $\hat{R}\otimes_R M|\hat{R}\otimes_R N$. An application of Theorem II gives the desired result: $M|N$. Q.E.D.

References

1. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. **97** (1960), 1–24. MR **22** #8034.

2. H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. No. 22 (1964), 5–60. MR **30** #4805.

3. ———, *Algebraic K-theory*, Benjamin, New York, 1968. MR **40** #2736.

4. A. Białynicki-Birula, *On the equivalence of integral representations of groups*, Proc. Amer. Math. Soc. **26** (1970), 371–377.

5. N. Bourbaki, *Algèbre*, Actualités Sci. Indust., no. 1272, Hermann, Paris, 1959. MR **21** #6384.

6. ———, *Algèbre commutative*. Chap. 3, Actualités Sci. Indust., no. 1293, Hermann, Paris, 1961. MR **30** #2027.

7. A. Dress, *On the decomposition of modules*, Bull. Amer. Math. Soc. **75** (1969), 984–986. MR **39** #5544.

8. A. Grothendieck, *Éléments de géométrie algébrique*. IV. *Étude locale des schémas et des morphismes de schémas*. II, Inst. Hautes Études Sci. Publ. Math. No. 24 (1965), 231 pp. MR **33** #7330.

9. R. G. Swan, *The number of generators of a module*, Math. Z. **102** (1967), 318–322. MR **36** #1434.

Mathematische Fakultät, Universität Bielefeld, 48 Bielefeld, Kurt-Schumacherstrasse 6, West Germany