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PRODUCTS OF UNCOUNTABLY MANY k-SPACES

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ABSTRACT. It is shown that if a product of nonempty spaces is a k-space then for each infinite cardinal $\mathfrak{n}$ some product of all but $\mathfrak{n}$ of the factors has each $\mathfrak{n}$-fold subproduct $\mathfrak{n}$-$\aleph_0$-compact (each $\mathfrak{n}$-fold open cover has a finite subcover). An example is given, for each regular $\mathfrak{n}$, of a space X which is not $\mathfrak{n}$-$\aleph_0$-compact (so $X^{\mathfrak{n}^+}$ is not a k-space) for which $X^{\mathfrak{n}}$ is a k-space.

1. **Introduction.** A subset F of a topological space X is *k-closed if* $F \cap K$ is closed in K for each compact subset K of X. A space in which each k-closed subset is closed is called a *k-space.* (No separation axioms will be assumed, so this definition differs from some of the other published definitions.) Although conditions under which finite or countable products of k-spaces will be k-spaces have been extensively studied, for instance in [**1**], [**2**], [**4**], [**6**], and [**7**], the only noteworthy results concerning products of k-spaces having uncountably many factors are included in the fact, proved in [**5**], that for a product of nonempty T_1-spaces to be a k-space, some product of all but countably many of its factors must be countably compact. We improve and extend this result with:

THEOREM. *If a product of nonempty spaces is a k-space then, for each infinite cardinal* $\mathfrak{n}$, *some product of all but* $\mathfrak{n}$ *of its factors has each* $\mathfrak{n}$*-fold subproduct* $\mathfrak{n}$-$\aleph_0$-compact.

Recall that a space is $\mathfrak{n}$-$\aleph_0$-compact if each $\mathfrak{n}$-fold open cover contains a finite subcover. As an immediate consequence of this theorem (together with Tychonoff's Theorem) we have:

COROLLARY. *All powers of a space X are k-spaces if and only if X is compact.*

It is amusing to contrast this result with the fact, established in [**8**], that all powers of a T_1-space X are normal if and only if X is compact. (Thus all powers of a T_1-space X are k-spaces if and only if all powers of

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X are normal.) The strength of our theorem on k-spaces is indicated by the following:

EXAMPLE. For each regular cardinal $\mathfrak{n}$ there exists a space X such that $X^{\mathfrak{n}}$ is a k-space but $X^{\mathfrak{m}}$ is not a k-space for any larger cardinal $\mathfrak{m}$.

Indeed, X can be taken to be $\mathfrak{n}$. (As usual, a cardinal $\mathfrak{n}$ is identified with the smallest ordinal of cardinality $\mathfrak{n}$ and, unless otherwise indicated, is assumed to bear the order topology.) This space X is certainly not $\mathfrak{n}$-$\aleph_0$-compact so, by the Theorem, $X^{\mathfrak{m}}$ is not a k-space for any $\mathfrak{m}$ greater than $\mathfrak{n}$. That $X^{\mathfrak{n}}$ is a k-space will follow from the more general considerations below.

Call a space $\mathfrak{n}$-*determined* if a subset is closed whenever it meets each subset S having $\mathfrak{n}$ or fewer elements in a set which is closed in S. Recall that a space is $\mathfrak{n}$-*bounded* if each subset with $\mathfrak{n}$ or fewer elements is contained in a compact set. Clearly $\mathfrak{n}$-boundedness is preserved by arbitrary products and each $\mathfrak{n}$-bounded $\mathfrak{n}$-determined space is a k-space.

PROPOSITION 1. *For $\mathfrak{n}$ an infinite cardinal, an $\mathfrak{m}$-fold product of $\mathfrak{n}$-determined spaces is $\mathfrak{n}$-determined if and only if all but at most $\mathfrak{n}$ of the factors are indiscrete.*

We call a space $<\mathfrak{n}$-*bounded* if each subset with fewer than $\mathfrak{n}$ elements is contained in a compact set and we call a space $<\mathfrak{n}$-*determined* if a subset is closed whenever it meets each subset S having fewer than $\mathfrak{n}$ elements in a set which is closed in S. Note that if $X=\mathfrak{n}$ and $\mathfrak{n}$ is regular, then X is $<\mathfrak{n}$-bounded and $<\mathfrak{n}$-determined. Thus our next result shows that, for this X, $X^{\mathfrak{n}}$ is a k-space.

PROPOSITION 2. *Let $X=\prod_{\alpha\in\mathfrak{n}} X_\alpha$. If each X_α is $<\mathfrak{n}$-bounded and $<\mathfrak{n}$-determined, then X is a k-space.*

2. **Proofs.**

PROOF OF THE THEOREM. The proof is by induction on $\mathfrak{n}$, so suppose that the Theorem holds for each cardinal less than $\mathfrak{n}$ and that $X=\prod_{\alpha\in\mathfrak{n}^+} X_\alpha$ is a nonempty k-space. In order to show that some product of all but at most $\mathfrak{n}$ of the factors of X has each $\mathfrak{n}$-fold subproduct $\mathfrak{n}$-$\aleph_0$-compact it suffices, by [**5**, Theorem 1], to show that all but $\mathfrak{n}$ of them must be $\mathfrak{n}$-$\aleph_0$-compact. Suppose that this is not the case; since by the induction hypothesis all but at most $\mathfrak{m}$ of the factors are $\mathfrak{m}$-$\aleph_0$-compact for each $\mathfrak{m}$ less than $\mathfrak{n}$, we may suppose that each X_α has an $\mathfrak{n}$-fold open cover which has no subcover of smaller cardinality. Passing to complements of unions, each X_α thus contains a nested family $\{A_\alpha^\lambda:\lambda\in\mathfrak{n}\}$ of nonempty closed sets with $\bigcap\{A_\alpha^\lambda:\lambda\in\mathfrak{n}\}=\varnothing$. Further, we may suppose that for each α there exists a point y_α in $X\backslash A_\alpha^0$.

For each λ in $\mathfrak{n}$ let B_λ be the union, over all γ in $\mathfrak{n}^+$, of the product sets whose αth factor is $\{y_\alpha\}$ for $\gamma \leqq \alpha \leqq \gamma+\lambda$ and A_α^λ otherwise. Let C_λ be the closure of $\bigcup_{\beta \leqq \lambda} B_\beta$ and set $C=\bigcup_{\lambda \in \mathfrak{n}} C_\lambda$; we will show that C is k-closed but not closed.

To see that C is not closed, note that since any finite subset of $\mathfrak{n}^+$ is contained in a segment $[\gamma, \gamma+\lambda]$ for some γ and λ, the point $y=(y_\alpha)$ is in the closure of C. On the other hand, y is not in C since, for λ in $\mathfrak{n}$, $(X_0 \backslash A_0^\lambda) \times (X_{\lambda+1} \backslash A_{\lambda+1}^\lambda) \times \prod_{\alpha \neq 0; \alpha \neq \lambda+1} X_\alpha$ is a neighborhood of y which does not meet $\bigcup_{\beta \leqq \lambda} B_\beta$, so y is not in the closure of C_λ. Now let $K \subseteq \prod_\alpha X_\alpha$ be compact, say $K=\prod_\alpha K_\alpha$. We show that $K \cap C$ is closed by showing $K \cap C = K \cap C_\lambda$ for some λ—since C_λ is closed, this suffices. For each α, note that K_α cannot meet cofinally many of the decreasing family $\{A_\alpha^\lambda : \lambda \in \mathfrak{n}\}$ since its intersection is empty. Thus there exists a $\lambda(\alpha)$ in $\mathfrak{n}$ such that $K_\alpha \cap A_\alpha^\lambda = \varnothing$ for each $\lambda > \lambda(\alpha)$. Since the domain of λ is $\mathfrak{n}^+$ while its range is $\mathfrak{n}$, there exists a λ_0 in $\mathfrak{n}$ such that $\{\alpha : \lambda(\alpha)=\lambda_0\}$ has cardinality $\mathfrak{n}^+$. For $\lambda > \lambda_0$ $K \cap C_\lambda = K \cap C_{\lambda_0}$ since for each point x in the closure of $\bigcup \{B_\beta : \lambda_0 < \beta \leqq \lambda\}$, x_α is in $A_\alpha^{\lambda_0+1}$ with fewer than $\mathfrak{n}$ exceptions. Consequently $K \cap C = K \cap C_{\lambda_0}$, so $K \cap C$ is closed. This contradicts the hypothesis that $\prod_\alpha X_\alpha$ is a k-space and thus completes the proof.

The proof above is a generalization of the proof sketched in [**3**, Exercise 7-J]. The first observation of our next proof implies that each subspace of an $\mathfrak{n}$-determined space is $\mathfrak{n}$-determined.

Proof of Proposition 1. Let us first note that if X is $\mathfrak{n}$-determined and x is in the closure of a subset A of X, then x is in the closure of some $\mathfrak{n}$-fold or smaller subset of A: Since an $\mathfrak{n}$-fold union of sets of cardinality $\mathfrak{n}$ itself has cardinality $\mathfrak{n}$, the operator which adjoins to A the closures of all of its $\mathfrak{n}$-fold subsets is idempotent, and is therefore the closure operator. Now suppose that $X=\prod_{\alpha \in \mathfrak{n}} X_\alpha$ where each X_α is $\mathfrak{n}$-determined and let x be in the closure of a subset A of X. We will show that X is $\mathfrak{n}$-determined by showing that x is in the closure of some $\mathfrak{n}$-fold subset of A.

Let F be any finite subset of $\mathfrak{n}$. Since x is in the closure of A, $\pi_F(x)$ is in the closure of $\pi_F(A)$, and hence, for some $\mathfrak{n}$-fold or smaller subset A_F of A, $\pi_F(x)$ is in the closure of $\pi_F(A_F)$. Let $A' = \bigcup \{A_F : F \subseteq \mathfrak{n}\}$ is finite and note that the cardinality of A' is less than or equal to $\mathfrak{n}$. Since x is clearly in the closure of A', A' is as desired.

For the converse, suppose $X=\prod_{\alpha \in \mathfrak{n}^+} X_\alpha$ where each X_α contains a point x_α and a closed subset F_α with x_α not in F_α. Let x be the point (x_α) and let F be the set of points in X whose αth coordinates, with at most $\mathfrak{n}$ exceptions, lie in F_α. Clearly F meets each $\mathfrak{n}$-fold or smaller set in a closed set. Since x is in the closure of F but is not in F, F is not closed, so this shows that X is not $\mathfrak{n}$-determined.

Proof of Proposition 2. Let $A \subseteq X$ be k-closed and let x be any point in the closure of A. We will produce a subset A' of such that x is in the closure of A' and such that, for each α in $\mathfrak{n}$, $\pi_\alpha A'$ has cardinality less than $\mathfrak{n}$. Since each X_α is $<\mathfrak{n}$-bounded, each $\pi_\alpha A'$, and hence A' itself, is contained in a compact set. It follows that x must be in A and hence that X is a k-space, as desired.

Let π^α denote the projection from X to $X^\alpha=\prod_{\beta<\alpha} X_\beta$ and note that, since $\mathfrak{n}$ is regular, the proof of Proposition 1 adapts easily to show that X^α is $<\mathfrak{n}$-determined. We first show that, for each α, $\pi^\alpha(x)$ is in $\pi^\alpha(A)$. Certainly $\pi^\alpha(x)$ is in the closure of $\pi^\alpha(A)$ and hence, since X^α is $<\mathfrak{n}$-determined, $\pi^\alpha(x)$ is in the closure of $\pi^\alpha(B)$ for some subset B of A having fewer than $\mathfrak{n}$ elements. Since X is $<\mathfrak{n}$-bounded, B is contained in some compact set K. Let K_1 be the projection of K onto $\prod_{\beta\geqq\alpha} X_\beta$, and let $K_2=\pi^\alpha(K)\cup\{\pi^\alpha(x)\}$. Since A is k-closed, $A\cap K_1\times K_2$ is closed in $K_1\times K_2$ and therefore its projection onto K_2, which is just $\pi^\alpha(A)\cap K_2$, is closed in K_2. Since $\pi^\alpha(B)\subseteq\pi^\alpha(A)\cap K_2$ and $\pi^\alpha(x)$ is in the intersection of the closure of $\pi^\alpha(B)$ with K_2, it follows that $\pi^\alpha(x)$ is in $\pi^\alpha(A)$, as desired.

To construct the set A', choose, for each α, a point x^α in A such that $\pi^\alpha(x^\alpha)=\pi^\alpha(x)$ and let $A'=\{x^\alpha:\alpha\in\mathfrak{n}\}$. It is clear that A' has the desired properties, so the proof is complete.

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