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AN INDEX FOR SET-VALUED MAPS IN INFINITE-DIMENSIONAL SPACES

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ABSTRACT. Previous fixed point indexes defined for a set-valued map in an infinite-dimensional space have required the values of this map to be convex sets. The corresponding assumption of this paper is that the values be (co-)acyclic sets, i.e., that the reduced Alexander cohomology group of each of these sets be trivial in each dimension.

Other assumptions are that the space is locally convex and that the map is compact and upper semicontinuous with no fixed points on the boundary of its domain.

The index is defined, proved to be homotopy invariant, and proved to vanish in case there are no fixed points. The main methods used are finite-dimensional approximation and the Vietoris-Begle mapping theorem.

1. **Introduction.** Let L denote a Hausdorff locally convex linear topological space (this generality is required since the compactness condition made later is too strong in infinite-dimensional spaces unless weak topologies are used). Let B denote a convex open neighborhood of the origin 0, and suppose that $B \cap S$ is bounded for every finite-dimensional subspace S of L.

Let F be a multi-valued (or set-valued) map on $\bar{B}$, that is, for each $x \in \bar{B}$ let $F(x)$ be a nonempty subset of L. Assume that F is upper semi-continuous, that is, whenever generalized sequences x_α ($\alpha \in A$) and y_α ($\alpha \in A$) converge to x and y respectively with x_α, $x \in \bar{B}$, $y_\alpha \in F(x_\alpha)$ for every α in a directed set A, then $y \in F(x)$. (Thus, in particular, each set $F(x)$ is closed. If F happens to be single-valued, i.e., each $F(x)$ is a set consisting of a single point, this is just the assumption that F is continuous.) Assume that F is compact, i.e., that $\mathrm{cl}(F(\bar{B}))=\mathrm{cl}(\bigcup_{x\in \bar{B}} F(x))$ is compact. Assume that for each $x \in \bar{B}$ the set $F(x)$ is (co-)acyclic, i.e., for each integer $k \geqq 0$ we have $H^k(F(x); Z)=\{0\}$, where $H^k(F(x); Z)$ denotes the reduced Alexander cohomology group of $F(x)$ in dimension k with integer coefficients. (See E. Spanier [7, p. 240, p. 306 ff]. Spanier uses a bar and a tilde to distinguish this cohomology group. Our notation for the (unreduced) Alexander cohomology group will be $\bar{H}^k$, the same as his.)

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Finally, assume that $x \notin F(x)$ for every $x \in \partial B$, the boundary of B. These assumptions are to remain in force for the remainder of this paper.

Many of the details in definition of the fixed point index and proof of its properties are motivated by the works of A. Granas and J. W. Jaworowski [4] and Jaworowski [5] who established the corresponding results for finite-dimensional L. They use homology theory where this paper uses cohomology theory however, and their coefficient group is the rationals or the integrals mod m, $m \geqq 2$, whereas this paper uses the integers, the usual coefficient group for discussions of topological index.

A. Cellina [1], A. Cellina and A. Lasota [2], A. Granas [3], and T. Ma [6] have proved results related to those of this paper for infinite-dimensional L in case each set $F(x)$ is convex.[1]

2. **Definition of the index.** First we show that $C = \bigcup_{x \in \partial B} (x - F(x))$ is a closed set. Let $c_\alpha = x_\alpha - y_\alpha$ ($\alpha \in A$) be a generalized sequence converging to c, where $x_\alpha \in \partial B$, $y_\alpha \in F(x_\alpha)$ for each α in some directed set A. Since $\mathrm{cl}(F(\bar{B}))$ is compact we may assume (by selecting a generalized subsequence if necessary) that y_α converges to some point y. But then $x_\alpha = c_\alpha + y_\alpha$ converges to $x = c + y$. By the upper semicontinuity of F, $y \in F(x)$ and thus $c = x - y \in C$, so C is closed.

The origin $0 \notin C$ since $x \notin F(x)$ for any $x \in \partial B$. Let K be any open symmetric (i.e., $K = -K$) convex neighborhood of 0 which is disjoint from C. Let S be any finite-dimensional subspace of L of dimension $k \geqq 2$ such that $\mathrm{cl}(F(\bar{B})) \subseteq S + K = \{s + k;\, s \in S,\, k \in K\}$. Finally, let $P: \mathrm{cl}(F(\bar{B})) \to S$ be continuous with $Py - y \in K$ for every $y \in \mathrm{cl}(F(\bar{B}))$.

For any $X \subseteq \bar{B}$, let $\mathscr{G}(X) = \{(x, y) \in L \times L;\, x \in X \text{ and } y \in F(x)\}$. Let $B \cap S = B'$ and $\partial B \cap S = \partial B'$. From the upper semicontinuity of F it is immediate that $\mathscr{G}(\partial B')$ is closed in $L \times L$. Since $\partial B'$ and $\mathrm{cl}(F(\bar{B}))$ are compact, it follows that $\mathscr{G}(\partial B')$ is compact.

In the space $L \times L$ let p_1 and p_2 be the projections onto the first and second factors respectively. The map p_1 has a restriction $p: \mathscr{G}(\partial B') \to \partial B'$. Clearly p is onto, and since $\mathscr{G}(\partial B')$ is compact, clearly p is a closed map. For each $x \in \partial B'$ and each integer $n \geqq 0$, $H^n(p^{-1}(x); Z) = H^n(\{x\} \times F(x); Z) \cong H^n(F(x); Z) = \{0\}$ by assumption. Therefore by the Vietoris-Begle mapping theorem [7, p. 344] the induced mapping $p^*: \bar{H}^n(\partial B'; Z) \to \bar{H}^n(\mathscr{G}(\partial B'); Z)$ is an isomorphism for each integer $n \geqq 0$. The symbol Z will be omitted for the remainder of the paper with the understanding that all cohomology groups are to be taken with integer coefficients.

Consider the map $c: \mathscr{G}(\bar{B}) \to L$ defined by $c(x, y) = p_1(x, y) - P(p_2(x, y)) = x - Py$ for each $(x, y) \in \mathscr{G}(\bar{B})$. Clearly c is continuous. When $(x, y) \in \mathscr{G}(\bar{B})$ with $x \in \partial B$ we have $y \in F(x)$ and $c(x, y) = x - Py = (x - y) - (Py - y)$. Since

[1] The author wishes to thank Dr. C. Rhee for his seminar which introduced the author to this literature.

$x-y\in C$ while $Py-y\in K$ we have $c(x, y)\neq 0$. Therefore c has a restriction $r:\mathcal{G}(\partial B')\to S\sim\{0\}$ and thus $r^*:\bar{H}^n(S\sim\{0\})\to\bar{H}^n(\mathcal{G}(\partial B'))$.

Let ρ be the usual retraction of $L\sim\{0\}$ onto ∂B. Let $h:S\sim\{0\}\to\partial B'$ be the restriction of ρ. Then $h^*:\bar{H}^n(\partial B')\to\bar{H}^n(S\sim\{0\})$.

The group $\bar{H}^{k-1}(\partial B')$ is free with one generator, where k is the dimension of S. Let u_0 be a generator. Then $(p^*)^{-1}r^*h^*u_0$ is in $\bar{H}^{k-1}(\partial B')$ and must therefore be some integer multiple i of $u_0:(p^*)^{-1}r^*h^*u_0=iu_0$. We define this integer $i=i[F; K, P, S]$ to be the fixed point index of F.

3. **Proof that the index is well defined.** First it must be shown that there are K, S, and P as required above. Then it must be shown that the index does not depend on the particular choice of these objects.

The existence of a set K as required follows from the local convexity of L. Since the open covering $\{x+K; x\in\mathrm{cl}(F(\bar{B}))\}$ of the compact set $\mathrm{cl}(F(\bar{B}))$ has a finite subcovering $x_1+K, \cdots, x_n+K$, we may take S to be the linear span of $x_1, \cdots, x_n$, enlarging S if necessary to insure that its dimension $k\geqq 2$.

For any $x\in L$, let $f(x)=\inf\{m>0; x\in mK\}$ and let $r(u)=1-u$ for $0\leqq u\leqq 1$ with $r(u)=0$ for $1\leqq u$. Then $r(f(x))$ is a continuous function on L which is 0 if and only if $x\notin K$. Then define

$$Py = \sum_{i=1}^{n} rf(y-x_i)x_i \Big/ \sum_{i=1}^{n} rf(y-x_i).$$

For $y\in\mathrm{cl}(F(\bar{B}))$ the denominator is never zero and clearly $Py\in S$. Moreover

$$Py - y = \sum_{i=1}^{n} rf(y-x_i)(x_i-y) \Big/ \sum_{i=1}^{n} rf(y-x_i),$$

and unless $x_i-y\in K$, $rf(y-x_i)=0$. Thus $Py-y$ is a convex linear combination of elements of K and thus $Py-y\in K$. This completes the proof that it is always possible to find suitable K, S, and P.

Let $\mathcal{S}$ be a subspace of L which contains S and has dimension $k+1$. We will now prove the crucial fact that $i[F; K, P, S]=i[F; K, P, \mathcal{S}]$. Let $\mathcal{B}=B\cap\mathcal{S}$ and $\partial\mathcal{B}=\partial B\cap\mathcal{S}$. Let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the two closed half-spaces in $\mathcal{S}$ determined by S. Then define $\varepsilon^+=\partial\mathcal{B}\cap\mathcal{S}^+$ and $\varepsilon^-=\partial\mathcal{B}\cap\mathcal{S}^-$.

Consider the commuting diagram

$$\begin{tikzcd}
\bar{H}^{k-1}(\partial B') \arrow[r] \arrow[d] & \bar{H}^{k-1}(\mathcal{G}(\partial B')) \arrow[d] & \bar{H}^{k-1}(S\sim\{0\}) \arrow[l] \arrow[d] & \bar{H}^{k-1}(\partial B') \arrow[l] \arrow[d] \\
\bar{H}^{k}(\varepsilon^-, \partial B') \arrow[r] & \bar{H}^{k}(\mathcal{G}(\varepsilon^-), \mathcal{G}(\partial B')) & \bar{H}^{k}(\mathcal{S}^-\sim\{0\}, S\sim\{0\}) \arrow[l] & \bar{H}^{k}(\varepsilon^-, \partial B') \arrow[l] \\
\bar{H}^{k}(\partial\mathcal{B}, \varepsilon^+) \arrow[r] \arrow[u] \arrow[d] & \bar{H}^{k}(\mathcal{G}(\partial\mathcal{B}), \mathcal{G}(\varepsilon^+)) \arrow[u] \arrow[d] & \bar{H}^{k}(\mathcal{S}\sim\{0\}, \mathcal{S}^+\sim\{0\}) \arrow[l] \arrow[u] \arrow[d] & \bar{H}^{k}(\partial\mathcal{B}, \varepsilon^+) \arrow[l] \arrow[u] \arrow[d] \\
\bar{H}^{k}(\partial\mathcal{B}) \arrow[r] & \bar{H}^{k}(\mathcal{G}(\partial\mathcal{B})) & \bar{H}^{k}(\mathcal{S}\sim\{0\}) \arrow[l] & \bar{H}^{k}(\partial\mathcal{B}) \arrow[l]
\end{tikzcd}$$

The vertical arrows from row one to row two and from row three to row four come from the exact sequence of a pair [7, p. 308]. The vertical arrows from row three to row two indicate homomorphisms coinduced by inclusion maps. The horizontal arrows from column one to column two, from column three to column two, and from column four to column three indicate homomorphisms coinduced by the appropriate restrictions of p_1, c, and ρ, respectively. Here the fact that $c(x, y)\neq 0$ for $x\in\partial B$, $y\in F(x)$ and the fact that $c(x, y)=x-Py\in x+S$ are crucial in verifying, for example, that $c(\mathscr{G}(\varepsilon^+))\subseteq\mathscr{S}^+\sim\{0\}$.

Standard considerations show that for $k\geqq 2$, $\bar{H}^{k-1}(\partial B')\to\bar{H}^k(\varepsilon^-, \partial B')$, $\bar{H}^k(\varepsilon^-, \partial B')\leftarrow\bar{H}^k(\partial\mathscr{B}, \varepsilon^+)$, and $\bar{H}^k(\partial\mathscr{B}, \varepsilon^+)\to\bar{H}^k(\partial\mathscr{B})$ are all isomorphisms.[2] The path from right to left along the top row of the diagram defines the index $i[F; K, P, S]$. An alternative path with the same endpoints travels down the right-hand side of the diagram (an isomorphism), along the bottom (a route defining the index $i[F; K, P, \mathscr{S}]$), and up along the left-hand side (the inverse of the previous isomorphism). Thus

$$i[F; K, P, \mathscr{S}] = i[F; K, P, S].$$

If now S_1 and S_2 are any finite-dimensional subspaces of L with $S_1\subseteq S_2$, we may construct a chain of subspaces between S_1 and S_2, each of codimension one in the next, and apply the above result several times to prove that $i[F; K, P, S_1]=i[F; K, P, S_2]$.

Now let $i[F; K_1, P_1, S_1]$ and $i[F; K_2, P_2, S_2]$ be any two indexes of F. Let $K=K_1\cap K_2$. Let S be any finite-dimensional subspace containing S_1 and S_2 such that $\mathrm{cl}(F(\bar{B}))\subseteq S+K$, and finally let a continuous $P:\mathrm{cl}(F(\bar{B}))\to S$ be chosen such that $Py-y\in K$ for every $y\in\mathrm{cl}(F(\bar{B}))$.

It is clear that $i[F; K, P, S]=i[F; K_1, P, S]$ and that $i[F; K_1, P_1, S]=i[F; K_1, P_1, S_1]$. We now show that $i[F; K_1, P, S]=i[F; K_1, P_1, S]$ proving that $i[F; K, P, S]=i[F; K_1, P_1, S_1]$. Similarly we will then have

$$i[F; K, P, S] = i[F; K_2, P_2, S_2]$$

and thus finally $i[F; K_1, P_1, S_1]=i[F; K_2, P_2, S_2]$. The indexes $i[F; K_1, P, S]$ and $i[F; K_1, P_1, S]$ are defined using the upper and lower paths of the diagram

$$\begin{tikzcd}
 & & \bar{H}^{k-1}(S\sim\{0\}) \arrow[dl, "r^*"'] & \\
\bar{H}^{k-1}(\partial B') \arrow[r, "p^*"] & \bar{H}^{k-1}(\mathscr{G}(\partial B')) & & \bar{H}^{k-1}(\partial B') \arrow[ul, "h^*"'] \arrow[dl, "h^*"] \\
 & & \bar{H}^{k-1}(S\sim\{0\}) \arrow[ul, "r_1^*"] &
\end{tikzcd}$$

[2] The author wishes to thank Dr. C. Doyle for his suggestion of the use of these isomorphisms to connect $\bar{H}^{k-1}(\partial B')$ and $\bar{H}^k(\partial\mathscr{B})$.

where the notation is as in §2 and $r_1:\mathscr{G}(\partial B')\to S\sim\{0\}$ is defined by $r_1(x,y)=x-P_1y$ for any $(x,y)\in\mathscr{G}(\partial B')$. But $r(x,y)=(x-y)+(y-Py)$ and $r_1(x,y)=(x-y)+(y-P_1y)$ are both in the set $(x-y)+K_1$ which does not contain 0. Therefore the standard linear homotopy between r and r_1 remains in $S\sim\{0\}$. Thus $r^*=r_1^*$ and the indexes are the same. Hereafter we denote the index of F by $i[F]$.

4. **Elementary properties of the index.** It is clear from the definition of $i[F]$ that it is an integer and depends only on the values of F on ∂B.

THEOREM 1. *If $i[F]\neq 0$, then there exists a fixed point $x_0\in B$, i.e., a point $x_0\in B$ for which $x_0\in F(x_0)$.*

PROOF. Assume that $x\notin F(x)$ for every $x\in B$. Then since $x\notin F(x)$ for every $x\in\partial B$, the set $R=\bigcup_{x\in\bar{B}}(x-F(x))$ is a closed set which does not contain the origin. Note that $C\subseteq R$. Let K be a symmetric convex open neighborhood of the origin which is disjoint from R. Construct S and P from K and F as in §3 above. Then consider

$$
\begin{tikzcd}
\bar{H}^{k-1}(\bar{B}\cap S) \arrow[r, "q^*"] \arrow[d, "i^*"] & \bar{H}^{k-1}(\mathscr{G}(\bar{B}\cap S)) \arrow[d, "j^*"] & \\
\bar{H}^{k-1}(\partial B\cap S) \arrow[r, "p^*"] & \bar{H}^{k-1}(\mathscr{G}(\partial B\cap S)) & \bar{H}^{k-1}(S\sim\{0\}) \arrow[ul, "r_1^*"'] \arrow[l, "r^*"] & \bar{H}^{k-1}(\partial B\cap S) \arrow[l, "h^*"']
\end{tikzcd}
$$

where q is the restriction of p_1 to $\mathscr{G}(\bar{B}\cap S)$, r_1 is the restriction of c to $\mathscr{G}(\bar{B}\cap S)$, and i and j are inclusions. By the Vietoris-Begle theorem q^* is an isomorphism (here we use for the only time the fact that F is (co-)acyclic-valued on B). The proof that $r_1(\mathscr{G}(\bar{B}\cap S))\subseteq S\sim\{0\}$ rests on the fact that $r_1(x,y)=x-Py=(x-y)-(Py-y)$ with $(x-y)\in R$ and $(Py-y)\in K$. By the commutativity of the diagram and the fact that $i^*=0$, it follows that $i[F]=0$.

DEFINITION. Suppose that $\Phi:\partial B\times[0,1]\to L$ is an upper semicontinuous, compact, multi-valued map with $\Phi(x,t)$ (co-)acyclic and $x\notin\Phi(x,t)$ for all $(x,t)\in\partial B\times[0,1]$. For $i=0,1$, let $F_i:\bar{B}\to L$ satisfy the conditions of section one for the map F, and suppose that $F_0(x)=\Phi(x,0)$ and $F_1(x)=\Phi(x,1)$ for every $x\in\partial B$. Whenever there exists a Φ as above we say that the boundary values of F_0 and F_1 are (co-)acyclically homotopic.

THEOREM 2. *If the boundary values of F_0 and F_1 are (co-)acyclically homotopic, then $i(F_0)=i(F_1)$.*

PROOF. The set $T=\bigcup_{(x,t)\in\partial B\times[0,1]}(x-\Phi(x,t))$ is closed and does not contain the origin so we may select a disjoint symmetric convex open neighborhood K of the origin. Select a finite-dimensional subspace S of L

with dimension $k \geqq 2$ such that $\mathrm{cl}(\Phi(\partial B, [0, 1])) \subseteq S + K$. Select a continuous $P\colon \mathrm{cl}(\Phi(\partial B, [0, 1])) \to S$ such that $Py - y \in K$ for every $y \in \mathrm{cl}(\Phi(\partial B, [0, 1]))$. Let

$$\mathscr{G} = \{(x, y, t) \in L \times L \times \boldsymbol{R};\, x \in \partial B \cap S,\, t \in [0, 1],\, y \in \Phi(x, t)\},$$
$$\mathscr{G}_0 = \{(x, y, t) \in L \times L \times \boldsymbol{R};\, x \in \partial B \cap S,\, t = 0,\, y \in \Phi(x, t)\},$$

and

$$\mathscr{G}_1 = \{(x, y, t) \in L \times L \times \boldsymbol{R};\, x \in \partial B \cap S,\, t = 1,\, y \in \Phi(x, t)\}.$$

Then consider the commutative diagram

$$\begin{tikzcd}
 & \bar{H}^{k-1}(\mathscr{G}_0) & & \\
\bar{H}^{k-1}(\partial B \cap S) \arrow[ur, "a_0^*"] \arrow[r, "a^*"] \arrow[dr, "a_1^*"'] & \bar{H}^{k-1}(\mathscr{G}) \arrow[u, "i^*"] \arrow[d, "j^*"] & \bar{H}^{k-1}(S \sim \{0\}) \arrow[l, "b^*"'] \arrow[ul, "b_0^*"'] \arrow[dl, "b_1^*"] & \bar{H}^{k-1}(\partial B \cap S) \arrow[l, "e^*"'] \\
 & \bar{H}^{k-1}(\mathscr{G}_1) & &
\end{tikzcd}$$

where a, a_0, and a_1 are restrictions of the projection of $L \times L \times \boldsymbol{R}$ onto its first factor, where b, b_0, and b_1 are restrictions of the map defined by $b(x, y, t) = x - Py$, where e is the usual retraction of $S \sim \{0\}$ onto $\partial B \cap S$, and where i and j are inclusions. For any $x \in \partial B \cap S$ the Vietoris-Begle theorem applies to the map $s\colon a^{-1}(x) \to [0, 1]$ defined by $s(x, y, t) = t$ for any $(x, y, t) \in a^{-1}(x)$. Thus $a^{-1}(x)$ is (co-)acyclic and the Vietoris-Begle mapping theorem applied to a proves that a^* is an isomorphism. Clearly $i(F_0)$ and $i(F_1)$ are defined using the upper and lower routes from right to left in the diagram above. Therefore $i(F_0) = i(F_1)$.

References

1. A. Cellina, *A theorem on the approximation of compact multi-valued mappings*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis.Mat. Natur. (8) **47** (1969), 429–433.

2. A. Cellina and A. Lasota, *A new approach to the definition of topological degree for multi-valued mappings*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis.Mat. Natur. (8) **47** (1969), 434–440.

3. A. Granas, *Theorem on antipodes and theorems on fixed points for a certain class of multi-valued mappings in Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 271–275. MR **22** #8365.

4. A. Granas and J. W. Jaworowski, *Some theorems on multi-valued mappings of subsets of the Euclidean space*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 277–283. MR **22** #11377.

5. J. W. Jaworowski, *Some consequences of the Vietoris mapping theorem*, Fund. Math. **45** (1958), 261–272; ibid. **46** (1959), 359. MR **24** #A2382.

6. Tsoy-Wo Ma, *Topological degrees of set-valued compact fields in locally convex spaces*, Thesis under Ky Fan, University of California, Santa Barbara, Calif., 1970, 77 pp.

7. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

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