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THE LIE ALGEBRA OF THE STRUCTURE GROUP OF A POWER-ASSOCIATIVE ALGEBRA

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ABSTRACT. For a strictly power-associative algebra A with identity let S be the span of the transitivity set of the identity under the action of the structure group. The main result of the paper is that the Lie algebra of the structure group is a subalgebra of the direct sum of the derivation algebra of A^+ and the space of left multiplications in A^+ by elements of S, and is equal to this sum if the characteristic is 0. It is also shown that S is a Jordan subalgebra of A^+.

1. The structure group, $\mathscr{G}(A)$, of a power-associative algebra A is defined to be the set of invertible linear transformations W such that there exists an invertible transformation $W^{\#}$ with $W(x)^{-1}=W^{\#-1}(x^{-1})$ for all x for which both sides are defined. It turns out to be an algebraic group, and therefore has a particular Lie algebra, $\mathscr{L}(\mathscr{G}(A))$, associated with it. The main purpose of this article is to determine, insofar as we can, this Lie algebra.

With the notations $X_1=\{W(1): W\in\mathscr{G}(A)\}$, $\langle X_1\rangle$ equal to the linear span of X_1 in A, and L_u^+ left multiplication by u in the algebra A^+, we prove the following:

THEOREM. *Let A be a finite-dimensional strictly power-associative algebra with an identity over an infinite field F of characteristic not* 2, *and $\mathscr{G}(A)$ the structure group of A. Then the Lie algebra $\mathscr{L}(\mathscr{G}(A))$ of the algebraic group satisfies*

$$\{L_u^+ : u \in \langle X_1\rangle\} \subseteq \mathscr{L}(\mathscr{G}(A)) \subseteq \{L_u^+;\ u\in\langle X_1\rangle\} \oplus \mathrm{Der}(A^+).$$

Moreover, if the characteristic is 0, *then* $\mathscr{L}(\mathscr{G}(A))=\{L_u^+ : u\in\langle X_1\rangle\}\oplus\mathrm{Der}(A^+)$.

One interesting consequence of this theorem is that $\langle X_1\rangle^+$ is a Jordan subalgebra of A^+. Also it enables us to prove that

$$\{L_u^+ : u\in\langle X_1\rangle\} \oplus \{D\in\mathrm{Der}(A^+) : D:\langle X_1\rangle\to\langle X_1\rangle\}$$

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is a Lie algebra for any characteristic $\neq 2$. If A is Jordan this reduces to $\{L_u : u \in A\} \oplus \mathrm{Der}(A)$, the Lie algebra which Braun and Koecher have associated with the structure group of a Jordan algebra [**1**, p. 289].

2. The proof involves much use of the differential calculus (cf. [**1**, Chapter 2] or [**3**, Chapter VI]). Following McCrimmon's notation in [**4**], for G a rational map, the linear map $\partial G|_x$ is the differential of G at x, and $\partial G|_x(u)$ will be written $\partial_u G|_x$. Then $x \to \partial_u G|_x$ is a rational map, and one has the usual rules for the differential of a product or composition of two functions, as well as for functions of two variables.

As an example, if A is a finite-dimensional algebra with identity over F, the generic minimal polynomial $m_x(\lambda) = \sum_{i=0}^{m} m_i(x)\lambda^i$ is defined and its coefficients m_i are polynomial functions (cf. [**3**, Chapter VI]). $N(x) = (-1)^m m_0(x)$ is called the generic norm.

$$0 = m_x(x) = N(x)1 + (-1)^m \left(\sum_{i=1}^{m} m_i(x) x^{i-1} \right) x$$

then shows that $t \to t^{-1}$ is a rational map defined on the Zariski dense set $X(N) = \{x : N(x) \neq 0\}$. Finally, $H_x = -\partial t^{-1}|_x$ is defined for all $x \in X(N)$.

McCrimmon has shown [**4**] that an invertible transformation W is in $\mathscr{G}(A)$ if and only if $W^{\#} H_{Wx} W = H_x$ for all $x \in X(N)$. For $x = 1$, it follows that if $W^{\#}$ exists, $W^{\#-1} = H_{W1} W$, so $W \in \mathscr{G}(A)$ implies $H_{Wx} W - H_{W1} W H_x = 0$ for all $x \in X(N)$. By examining H_x, it is clear that $x \to N(x)^2 H_x$ can be thought of as a polynomial map defined for all x. Thus $W \in \mathscr{G}(A)$ implies W is a zero of the polynomial map $W \to N(Wx)^2 N(W1)^2 \{H_{Wx} W - H_{W1} W H_x\}$ for each $x \in X(N)$. However McCrimmon has also proven that $W \in \mathscr{G}(A)$ implies $N(Wx) = N(W1)N(x)$ holds for all x in A [**4**, p. 543]. Thus any $W \in \mathscr{G}(A)$ is a zero of the polynomial maps

$$\mathscr{A}_x(W) = N(x)^2 N(W1)^2 \{H_{Wx} W - H_{W1} W H_x\}, \qquad x \in X(N),$$
$$\beta_x(W) = N(Wx) - N(W1)N(x), \qquad x \in A.$$

Conversely, suppose W is invertible and is a zero of the $\mathscr{A}_x$, $x \in X(N)$, and β_x, $x \in A$. Then $N(W1) = 0$ would force $N(Wx) = 0$ for all $x \in A$, whence W could not be invertible. Hence $N(W1) \neq 0$, so $\mathscr{A}_x(W) = 0$ for $x \in X(N)$ implies $H_{Wx} W - H_{W1} W H_x = 0$ for $x \in X(N)$ and W is in $\mathscr{G}(A)$. Thus $\mathscr{G}(A)$ is an algebraic group defined by the $\mathscr{A}_x$, $x \in X(N)$ and the β_x, $x \in A$.

Now if $\mathscr{A}_{xij}(W) = (\mathscr{A}_x(W))_{i,j}$ with respect to a fixed matrix representation of $\mathrm{Hom}(A, A)$, then $\mathscr{A}_{xij}(W) = 0$ for all i, j if and only if $\mathscr{A}_x(W) = 0$. Thus in particular, $\{\mathscr{A}_{xij}\} \subseteq \mathscr{I}(\mathscr{G}(A))$, the ideal of all polynomial functions which are zeroed by all $W \in \mathscr{G}(A)$. Since $\mathscr{L}(\mathscr{G}(A))$ may be thought of as

$$\{T \in \mathrm{Hom}(A, A) : \partial_T \mathscr{A}|_I = 0 \text{ for all } \mathscr{A} \in \mathscr{I}(\mathscr{G}(A))\}$$

(cf. [2, p. 128]), it follows that $T\in\mathscr{L}(\mathscr{G}(A))$ implies $\partial_T\mathscr{A}_{xij}|_I=0$ for all $x\in X(N)$ and all i,j. But this is equivalent to $\partial_T\mathscr{A}_x|_I=0$ for all $x\in X(N)$.

Now we have

LEMMA 1. *$T\in\mathscr{L}(\mathscr{G}(A))$ implies $\partial_{Tx}H_t|_x=-2L^+_{T1}H_x+[T,H_x]$ for all x in $X(N)$.*

PROOF.

$$\begin{aligned}\partial_T\mathscr{A}_x|_I &= \partial_T\{N(x)^2N(W1)^2\}|_I\{H_{Ix}I - H_{I1}IH_x\}\\ &\quad + N(x)^2N(I1)^2\partial_T(H_{Wx}W - H_{W1}WH_x)|_I\\ &= N(x)^2\partial_T(H_{Wx}W - H_{W1}WH_x)|_I.\end{aligned}$$

Thus $T\in\mathscr{L}(\mathscr{G}(A))$ implies

$$\begin{aligned}0 &= \partial_T(H_{Wx}W - H_{W1}WH_x)|_I\\ &= \partial_TH_{Wx}|_II + H_{Ix}\partial_TW|_I - \partial_TH_{W1}|_IIH_x\\ &\quad - H_{I1}\partial_TW|_IH_x - H_{I1}I\partial_TH_x|_I.\end{aligned}$$

The chain rule gives $\partial_TH_{Wx}|_I=\partial_{Tx}H_t|_x$ and $\partial_TH_{W1}|_I=\partial_{T1}H_t|_1$, and the latter becomes $-2L^+_{T1}$ by the identity $\partial_uH_t|_1=-2L^+_u$ which McCrimmon proves in [4]. Since $W\to W$ is linear, $\partial_TW|_I=T$, and finally, $\partial_TH_x|_I=0$. Thus $0=\partial_{Tx}H_t|_x+H_xT+2L^+_{T1}H_x-TH_x$, which completes the proof.

LEMMA 2. *If $T\in\mathscr{L}(\mathscr{G}(A))$ and $T1=0$, then T is in* $\mathrm{Der}(A^+)$.

PROOF. $T1=0$ in Lemma 1 implies $\partial_{Tx}H_t|_x=[T,H_x]$ for all x in $X(N)$. We apply $\partial_u|_1$ to this as a function of x.

$$\begin{aligned}\partial_u\{\partial_{Tx}H_t|_x\}|_1 &= \partial_u\{\partial_{T1}H_t|_x\}|_1 + \partial_u\{\partial_{Tx}H_t|_1\}|_1\\ &= \partial_u(-2L^+_{Tx})|_1 = -2L^+_{Tu}.\end{aligned}$$

On the other hand, $W\to[T,W]$ is linear, so the chain rule gives $\partial_u[T,H_x]|_1=[T,\partial_uH_t|_1]=[T,-2L^+_u]$. Since $X(N)$ is dense, we have $L^+_{Tu}=[T,L^+_u]$ for all u in A and T is in $\mathrm{Der}(A^+)$.

LEMMA 3. *L^+_u is in $\mathscr{L}(\mathscr{G}(A))$ if u is in $\langle X_1\rangle$.*

PROOF. If x is in X_1, then $x=W1$ for some W in $\mathscr{G}(A)$, whence $W^\#H_{W1}W=H_1=I$ implies $H_x=H_{W1}$ is in $\mathscr{G}(A)$. Thus if $\mathscr{A}$ is in $\mathscr{I}(\mathscr{G}(A))$, $\mathscr{A}(H_x)=0$ for all x in X_1. But applying $\partial_u|_1$ to $0=\mathscr{A}(H_x)$ yields $\partial_{L_u^+}\mathscr{A}|_I=0$ for all u in $\langle X_1\rangle$ and hence L^+_u is in $\mathscr{L}(\mathscr{G}(A))$.

LEMMA 4. *If T is in $\mathscr{L}(\mathscr{G}(A))$, u is in $\langle X_1\rangle$, then $T(u)$ is in $\langle X_1\rangle$.*

PROOF. Clearly $\mathscr{G}(A):X_1\to X_1$ and, by [2, p. 135], $\mathscr{L}(\mathscr{G}(A))$ is contained in the linear span of $\mathscr{G}(A)$ in $\mathrm{Hom}(A,A)$.

Now we have: if T is in $\mathscr{L}(\mathscr{G}(A))$ then $T1$ is in $\langle X_1\rangle$ and so L^+_{T1} is in $\mathscr{L}(\mathscr{G}(A))$. Then also $T-L^+_{T1}$ is in $\mathscr{L}(\mathscr{G}(A))$ and so by Lemma 2 is in $\mathrm{Der}(A^+)$. Consequently $T\in\mathscr{L}(\mathscr{G}(A))$ implies $T=L^+_{T1}+(T-L^+_{T1})$ is in $\{L^+_u: u\in\langle X_1\rangle\}+\mathrm{Der}(A^+)$. Together with Lemma 3, this implies the first part of the theorem. To complete the proof of the theorem, we shall suppose the characteristic is 0 and that D is in $\mathrm{Der}(A^+)$. Then (cf. [**2**, p. 143]) D is in $\mathscr{L}(\mathrm{Aut}(A^+))$, the Lie algebra associated with the algebraic group of automorphisms of A^+. However, $\mathrm{Aut}(A^+)\subseteq\mathscr{G}(A)$ is clear, since if W is in $\mathrm{Aut}(A^+)$, $1=W(xx^{-1})=W(x)W(x^{-1})$ and $W(x)^{-1}=(W^{-1})^{-1}(x^{-1})$, and one has $W^{\#}=W^{-1}$, with $W\in\mathscr{G}(A)$. Consequently $\mathscr{I}(\mathrm{Aut}(A^+))\supseteq\mathscr{I}(\mathscr{G}(A))$. Hence if $\partial_D\mathscr{A}|_I=0$ for all $\mathscr{A}\in\mathscr{I}(\mathrm{Aut}(A^+))$, certainly $\partial_D\mathscr{A}|_I=0$ for all $\mathscr{A}\in\mathscr{I}(\mathscr{G}(A))$ and D is in $\mathscr{L}(\mathscr{G}(A))$.

COROLLARY 1. *$\langle X_1\rangle^+$ is a Jordan subalgebra of A^+.*

PROOF. Lemmas 3 and 4 imply that $\langle X_1\rangle^+$ is a subalgebra of A^+. Now for any $x\in X_1$, we have seen that H_x is in $\mathscr{G}(A)$, so $(H_xy)^{-1}=H_x^{-1}(y^{-1})$ for all $y\in X(N)$. But (cf. [**4**, p. 544]) applying $\partial_u|_1$ to this as a function in x yields $H_yL^+_y|_{\langle X_1\rangle}=L^+_yH_y|_{\langle X_1\rangle}$ for all $y\in X(N)$, and in particular for invertible y in $\langle X_1\rangle$. However this implies that $\langle X_1\rangle^+$ is Jordan by McCrimmon's Equivalence Theorem in [**4**].

COROLLARY 2. *$\{L^+_u$: $u\in\langle X_1\rangle\}\oplus\{D\in\mathrm{Der}(A^+)$: D: $\langle X_1\rangle\to\langle X_1\rangle\}$ is a Lie algebra.*

PROOF. (i) For u, $v\in\langle X_1\rangle$, $[L^+_u, L^+_v]\in\mathrm{Der}(A^+)$ by the theorem, since both L^+_u and L^+_v are in $\mathscr{L}(\mathscr{G}(A))$. Also $[L^+_u, L^+_v]:\langle X_1\rangle\to\langle X_1\rangle$ as $\langle X_1\rangle^+$ is a subalgebra.

(ii) For $u\in\langle X_1\rangle$ and D a derivation which maps $\langle X_1\rangle$ into itself, clearly $[L^+_u, D]=L^+_{Du}\in\{L^+_v: v\in\langle X_1\rangle\}$.

(iii) The subset of derivations which map $\langle X_1\rangle$ into itself is a Lie subalgebra.

3. We conclude with an example (cf. [**2**, p. 143]) of a Jordan algebra A with $\mathscr{L}(\mathscr{G}(A))=\{L_a: a\in A\}\subsetneq\{L_a: a\in A\}\oplus\mathrm{Der}(A)$.

Suppose F is a field of characteristic p which is not perfect and let $\beta\in F$, $\beta^{1/p}\notin F$. Then $A=F(\beta^{1/p})$ is a field and $\mathrm{Aut}(A)$ consists of only the identity automorphism. Clearly $\{L_a: 0\neq a\in A\}$ is in $\mathscr{G}(A)$. But then $W\in\mathscr{G}(A)$ implies $L_{(W1)^{-1}}W$ is also in $\mathscr{G}(A)$, and maps 1 to 1. Consequently $L_{(W1)^{-1}}W$ is in $\mathrm{Aut}(A)$ (cf. [**1**, p. 156]), which finally implies $W=L_{W1}$. That is, $\mathscr{G}(A)=\{L_a: 0\neq a\in A\}$. Consequently $\mathscr{L}(\mathscr{G}(A))$, which is contained in the linear span of $\mathscr{G}(A)$ in $\mathrm{Hom}(A, A)$, is contained in $\{L_a: a\in A\}$. On the other hand, A has derivations which are certainly not contained in $\{L_a: a\in A\}$.

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