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ELEMENTARY EXTENSIONS OF LINEAR TOPOLOGICAL ABELIAN GROUPS

R. G. PHILLIPS AND P. L. SPERRY

ABSTRACT. R. MacDowell and E. Specker obtain a structure theorem for elementary extensions of the integers by considering a certain residue mapping. In this paper we characterize those abelian groups in which an analogous situation exists and obtain the MacDowell-Specker result as a special case of our theory.

Introduction. In [3] MacDowell and Specker obtain a structure theorem for elementary extensions $^*Z$ of Z (Z is the additive group of integers). Their method is to construct a homomorphism from $^*Z$ to the Z-adic completion $\hat{Z}$ of Z which extends the natural embedding μ of Z into $\hat{Z}$. It turns out that the kernel of this homomorphism is the divisible subgroup d^*Z of $^*Z$ so that $^*Z=d^*Z\oplus K$ where K is a subgroup containing Z which is isomorphic to a subgroup of $\hat{Z}$ by an isomorphism which extends μ. The homomorphism constructed in [3] essentially sends an element of $^*Z$ into its sequence of standard residues modulo n for $n\in N$ (N is the set of positive natural numbers).

In this paper, we generalize the methods of [3] in the following way. Suppose G is an abelian linear topological group, that is, its topological structure is defined by a filter of subgroups $\mathscr{D}$. Then there are elementary extensions $^*G$ of G which extend the topological structure of G. We investigate the existence of homomorphisms on these $^*G$ into the $\mathscr{D}$ completion $\hat{G}$ of G which extend the canonical map of G into $\hat{G}$ and which are, in some sense, "residue" mappings. According to our Theorem 1 such "residue" mappings exist for every such extension if and only if $\mathscr{D}$ generates a topology coarser than the finite index topology; the mapping defined in I of Theorem 1 is the analog of the mapping defined in [3]. As a corollary of Theorem 1 we obtain a generalization of the Structure Theorem in [3].

Definitions. All groups are abelian; a pair $(G,\mathscr{D})$ will denote a group G and a filter $\mathscr{D}$ of subgroups of G. Given $(G,\mathscr{D})$, G will carry the topology

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generated by $\mathscr{D}$ and $\mathscr{D}$ will be written as an indexed family $(H_i)_{i\in I}$; I will be partially ordered by: $i\leqq j$ iff $H_i\supset H_j$. Given $(G,\mathscr{D})$ the completion of G is the projective limit $\hat{G}$ of the system $(G/H_i, \Pi_{ij})$ where, if $i\leqq j$, $\Pi_{ij}(g_j+H_j)=g_j+H_i$; $\hat{G}$ has the relative product topology which has as a base for the neighborhoods of zero the filter $(\tilde{H}_i)_{i\in I}$ where $(g_i+H_i)_{i\in I}$ is in $\tilde{H}_k$ iff g_k is in H_k. The mapping μ of G into $\hat{G}$ such that $\mu(g)=(g+H_i)_{i\in I}$ is a continuous homomorphism of G onto a dense subgroup of the complete Hausdorff space $\hat{G}$ (see [**1**]).

Given $(G,\mathscr{D})$, $L(G,\mathscr{D})$ will denote a first order language which includes a distinct constant symbol for each element of G, a function symbol "+" for the group operation on G, and for each $H\in\mathscr{D}$ a predicate symbol $H(x)$ such that $H(a)$ holds in G iff $a\in H$. Let $K(G,\mathscr{D})$ denote all sentences of $L(G,\mathscr{D})$ which hold in G; a model of $K(G,\mathscr{D})$ will be called a $\mathscr{D}$-elementary extension of G. If $*G$ is a $\mathscr{D}$-elementary extension of G and $H\in\mathscr{D}$, $*H$ will denote the set of all elements $\mathscr{Y}$ of $*G$ which satisfy $H(\mathscr{Y})$ in $*G$. Note that G is a subgroup of $*G$ and $*H\cap G=H$ for all $H\in\mathscr{D}$. A $\mathscr{D}$-elementary extension $*G$ of G will be called a strong $\mathscr{D}$-elementary extension if for each element $(g_i+H_i)_{i\in I}$ of $\hat{G}$ there is an element x of $*G$ such that $x-g_i$ is an element of $*H_i$ for each i in I. The topology generated on $*G$ by $(*H_i)_{i\in I}$ is called the S-topology (see [**4**]).

LEMMA. *For any pair $(G,\mathscr{D})$ and any $\mathscr{D}$-elementary extension $*G$ of G there is a strong $\mathscr{D}$-elementary extension $**G$ of G which is also an elementary extension of $*G$.*

PROOF. Extend $L(G,\mathscr{D})$ to any first order language L' that includes a constant symbol for each element of $*G$. Let K' be the set of all sentences of L' which hold in $*G$. Extend L' to a first order language L'' by adding, for each $\alpha\in\hat{G}$, a distinct new constant a_α. For each $\alpha\in\hat{G}$ let $K_\alpha=\{H_i(a_\alpha-g_i):i\in I\}$ where $\alpha=(g_i+H_i)_{i\in I}$. Let $K_1=(\bigcup\{K_\alpha|\alpha\in\hat{G}\})\cup K'$. Now for each $(g_i+H_i)_{i\in I}$ in $\hat{G}$ and each finite subset J of I there is a g in G such that $g-g_i$ is in H_i for each i in J. From this it is routine to show that K_1 is consistent. Any model of K_1 is clearly a strong $\mathscr{D}$-elementary extension of G containing $*G$ as an elementary substructure.

THEOREM 1. *For any pair $(G,\mathscr{D})$ the following conditions are equivalent*:

I. *For every $\mathscr{D}$-elementary extension $*G$ of G there is a homomorphism ρ of $*G$ into $\Pi_I(G/H_i)$ which extends μ and has the property that $\rho(a)=(g_i+H_i)_{i\in I}$ implies $a-g_i\in *H_i$ for all $i\in I$.*

II. *For every $\mathscr{D}$-elementary extension $*G$ of G there is a homomorphism ρ of $*G$ into $\hat{G}$ which extends μ and satisfies* $\operatorname{Ker}\rho=\bigcap_I *H_i$ *and* $\rho(*H_i)\subset\tilde{H}_i$.

III. *G is dense in every strong $\mathscr{D}$-elementary extension $*G$ of G (with respect to the S-topology).*

IV. *G is dense in every $\mathscr{D}$-elementary extension $^*G$ of G (with respect to the S-topology).*

V. *G/H_i is finite for all $i\in I$.*

PROOF. We show I⇒II⇒III⇒IV⇒V⇒I.

I⇒II. Let $^*G$ be a $\mathscr{D}$-elementary extension of G and let ρ satisfy the property in I. To show that ρ has the property of II we only need prove that ρ maps $^*G$ into $\hat{G}$. Given $a\in {}^*G$ let $\rho(a)=(g_i+H_i)_{i\in I}$. Then if $J=\{i_1, \cdots, i_n\}$ is any finite subset of I and

$$X_J = \exists x[H_{i_1}(x - g_{i_1}) \wedge \cdots \wedge H_{i_n}(x - g_{i_n})]$$

then X_J holds in G since it holds in $^*G$. Thus there is a $g\in G$ so that $g-g_{i_t}\in H_{i_t}$ for $t=1, 2, \cdots, n$ and hence $\rho(a)\in\hat{G}$.

II⇒III. Let $^*G$ be a strong $\mathscr{D}$-elementary extension of G and let ρ satisfy the property in II. Given $a\in {}^*G$ and $k\in I$ there is an $h\in\tilde{H}_k$ and a $g\in G$ such that $\rho(a)+h=\mu(g)$ since $\mu(G)$ is dense in $\hat{G}$. If $h=(g_i+H_i)_{i\in I}$, there is a $b\in {}^*G$ such that $b-g_i\in {}^*H_i$ for all $i\in I$. Let $\rho(b)=(t_i+H_i)_{i\in I}$. Since $\rho(b-g_i)\in\tilde{H}_i$ for each $i\in I$ we have $g_i-t_i\in H_i$ for all i and hence $\rho(b)=h$. Since $h\in\tilde{H}_k$, $g_k\in H_k$ and hence $b\in {}^*H_k$; $a+b-g\in \operatorname{Ker}\rho=\bigcap_I {}^*H_i$. Therefore $a-g\in {}^*H_k$ which proves that G is dense in $^*G$.

III⇒IV. Follows immediately from the lemma.

IV⇒V. Suppose G/H is infinite for some $H\in\mathscr{D}$. We set $K'=K(G,\mathscr{D})\cup\{\neg H(\alpha-g):g\in G\}$ where α is a new constant symbol. K' is consistent since G serves as a model for any finite subset of K' with a suitable interpretation of α. But G is not dense in any model of K'.

V⇒I. Let $G/H_i=\{H_i, g_{i_1}+H_i, \cdots, g_{i_n}+H_i\}$. Then if

$$X_i = \forall x[H_i(x)\,\overline{\vee}\cdots\overline{\vee}\,H_i(x - g_{i_n})]$$

where "$\overline{\vee}$" denotes exclusive "or", X_i holds in G for all $i\in I$ and hence holds in every $\mathscr{D}$-elementary extension $^*G$ of G. Given such a $^*G$, if $a\in {}^*G$ we define $\rho(a)=(g_i+H_i)_{i\in I}$ if and only if $a-g_i\in {}^*H_i$ for $i\in I$. Conditions X_i assure that ρ is well defined and it follows easily that ρ satisfies the condition in I.

COROLLARY. *Let G be any group with $H_n=nG$ for $n\in N$. If for every elementary extension $^*G$ of G, $^*G=D\oplus K$ where D is divisible, $G\subset K$ and there is a homomorphism ϕ of K into $\hat{G}$ which extends μ, has its kernel contained in G and has a pure image, then* I–V *are also true.*

PROOF. The S-topology on any $^*G$ is the Z-adic topology. Since $\phi[K]$ is pure in $\hat{G}$, the relative and Z-adic topologies coincide on $\phi[K]$ so $\mu(G)$ is dense in $\phi[K]$ with respect to the Z-adic topology. Hence $\phi[K]/\mu[G]$ is divisible and isomorphic to K/G so that $*G/G$ is isomorphic to $D\oplus(K/G)$ which is divisible, hence G is dense in $*G$ and IV holds.

COROLLARY. *Let G be a reduced torsion free group and let $H_n=nG$ for $n\in N$. Then* I–V *are equivalent to the property that $*G=d*G\oplus K$ for every elementary extension $*G$ of G where $G\subset K$, $d*G$ is the divisible subgroup of $*G$ and K is isomorphic to a subgroup of $\hat{G}$ by an isomorphism which extends μ.*

PROOF. Assume II. The kernel of the ρ guaranteed by II is $\bigcap n*G=d*G$ since $*G$ is torsion free. Since G is reduced and pure in $*G$ there is a subgroup K of $*G$ which contains G such that $*G=d*G\oplus K$. It is easily seen that $\rho|K$ satisfies the conditions needed.

Conversely, suppose $\phi:K\to\hat{G}$ is an isomorphism which extends μ, let $\pi: *G\to K$ be the projection, then $\phi\pi$ is easily seen to satisfy II since $(nG)^{\hat{}}=n\hat{G}$ (see [**2**]).

Note (see for instance [**5**, Corollary 2]) that ΠJ_p, where J_p is the p-adic integers, is $\hat{Z}$ with respect to the Z-adic topology.

COROLLARY (MACDOWELL AND SPECKER). *Any elementary extension of Z is isomorphic to the direct sum of its divisible subgroup and a subgroup of ΠJ_p.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208