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STARLIKE MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper we study meromorphic univalent functions which map the unit disk onto the exterior of a domain which is starlike with respect to some finite point different from the origin. We obtain bounds on the arc length, an integral representation, and bounds on the maximum modulus of starlike meromorphic functions.

1. **Introduction.** Let $U(p)$ denote the class of univalent meromorphic functions $f(z)$ in the unit disk E with a simple pole at $z=p>0$ and with the normalization $f(0)=0$ and $f'(0)=1$. Let $U^*(p, w_0)$ be the subclass of $U(p)$ such that $f(z)\in U^*(p, w_0)$ if and only if there is a ρ, $p<\rho<1$, with the property that

$$\operatorname{Re}\{z[f'(z)/(f(z)-w_0)]\}<0$$

for $\rho<|z|<1$. The functions in $U^*(p, w_0)$ map $|z|<r<\rho$ (for some ρ, $p<\rho<1$) onto the complement of a set which is starlike with respect to w_0. Further the functions in $U^*(p, w_0)$ all omit the value w_0. This class of starlike meromorphic functions is developed from Robertson's concept of star center points [7].

For $f(z)\in U^*(p, w_0)$, there is a function $P(z)$ regular in E with $P(0)=1$ and $\operatorname{Re}\{P(z)\}\geqq 0$ such that

$$z\frac{f'(z)}{f(z)-w_0}+\frac{p}{z-p}-\frac{pz}{1-pz}=-P(z) \tag{1}$$

for all $z\in E$. Let $\Sigma^*(p, w_0)$ denote the class of functions $f(z)$ which satisfy (1) and the conditions $f(0)=0$, $f'(0)=1$. Then $U^*(p, w_0)$ is a subset of $\Sigma^*(p, w_0)$. For $p<2-\sqrt{3}$ it follows from the proof of Theorem 4 of [5] that $U^*(p, w_0)=\Sigma^*(p, w_0)$.

In this paper we shall use (1) to study the coefficient problem, to obtain an integral representation for $U^*(p, w_0)$, to obtain an estimate of the arc length of the image of $|z|=r$, and to obtain a bound on the modulus of the functions. This work was motivated by the results on regular starlike univalent functions by Keogh [3] and Pommerenke [6].

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2. **Coefficients and star center points.** Suppose we have a function $f(z)$ and a function $P(z)$ which satisfy (1) and have the expansions $f(z)=z+a_2z^2+\cdots$ for $|z|<p$ and $P(z)=1+b_1z+b_2z^2+\cdots$ for $|z|<1$. Then by taking the series expansion of both sides of (1) we have

$$b_1 = p + 1/p + 1/w_0 \text{ and} \tag{2}$$

$$b_2 = p^2 + 1/p^2 + 1/w_0^2 + 2a_2/w_0. \tag{3}$$

Thus, for a fixed w_0, equation (2) implies that the functions $P(z)$ which may be used in (1) have a fixed coefficient b_1. Let $\mathscr{P}(b_1)$ denote the class of functions $P(z)$ regular in E with the properties $P(0)=1$, $P'(0)=b_1$, and $\operatorname{Re}\{P(z)\}\geqq 0$.

From equation (2), and the fact that $|b_1|\leqq 2$, we have

$$p/(1+p)^2 \leqq |w_0| \leqq p/(1-p)^2 \tag{4}$$

and, since $p+1/p>2$, we have $\operatorname{Re} w_0<0$. For $p<2-\sqrt{3}$, the bounds on $|w_0|$ in (4) are sharp, because the function

$$f_0(z) = \frac{z}{1-(p+1/p)z+z^2}$$

maps $|z|<1$ onto the extended plane with a slit on the negative real axis from $-p/(1+p)^2$ to $-p/(1-p)^2$. Thus for $p<2-\sqrt{3}$ and $-p/(1-p)^2\leqq w_0\leqq -p/(1+p)^2$, we have $f_0(z)\in U^*(p, w_0)$.

By considering equation (3) we obtain the following theorem.

THEOREM 1. *If $f(z)\in\Sigma^*(p, w_0)$, then the second coefficient is given by*

$$a_2 = \tfrac{1}{2}w_0(b_2 - p^2 - 1/p^2 - 1/w_0^2) \tag{5}$$

where b_2 is the second coefficient of a function in $\mathscr{P}(b_1)$, that is the region of variability for a_2 is contained in the disk

$$|a_2 + \tfrac{1}{2}w_0(p^2 + 1/p^2 + 1/w_0)| \leqq |w_0|. \tag{6}$$

Further there is a p_0, $0.39<p_0<0.61$, such that if $p<p_0$, then $\operatorname{Re}\{a_2\}>0$.

PROOF. To establish the proof we need only see when $\operatorname{Re}\{a_2\}>0$. From (3) we have

$$\begin{aligned}\operatorname{Re}\{a_2\} &\geqq -|w_0| - \tfrac{1}{2}\operatorname{Re}\{1/w_0 + (p^2+1/p^2)w_0\} \\ &= -|w_0| - \tfrac{1}{2}\Big\{r_1\cos\theta - p - 1/p \\ &\qquad - p\left(p^2+\frac{1}{p^2}\right)\frac{1+p^2-pr_1\cos\theta}{(1+p^2)^2 - 2(1+p^2)pr_1\cos\theta + p^2r_1^2}\Big\},\end{aligned}$$

where $b_1=r_1e^{i\theta}$. The minimum of

$$\frac{a - r\cos\theta}{a^2 - 2ar\cos\theta + r^2}$$

occurs when $\theta=\pi$, and the max of $r_1\cos\theta$ occurs when $\theta=0$. Thus we have

$$\begin{aligned}\operatorname{Re}\{a_2\} &\geqq -\frac{p}{(1-p)^2} + \frac{(1-p)^2}{2p} \\ &\quad + \frac{p}{2}\left(p^2 + \frac{1}{p^2}\frac{1}{1+p_2+pr_1}\right) \\ &\geqq \frac{p^6 - 2p^5 - p^4 - p^2 - 2p + 1}{p(1-p)^2(1+p)^2}.\end{aligned}$$

Hence $\operatorname{Re}\{a_2\}>0$ for $p<p_1$, where p_1 is the smallest positive root of $p^6-2p^5-p^4-p^2-2p+1$. A calculation shows that $0.39<p_1<0.4$. By consideration of equation (5) it can be shown that $\operatorname{Re}\{a_2\}$ may be negative for $p>0.61$.

REMARKS. (1) By considering (5) we note that the maximum value of $|a_2|$ is $p+1/p$, which is the sharp upper bound for the second coefficient of functions in $U(p)$ [2]. However, the region of variability for a_2 for functions in $U^*(p, w_0)$ is probably not the complete disk (6).

(2) By continuing the expansion of (1) we have that a_n, $n>2$, is given by a formula similar to that for a_2. For instance

$$a_3 = \frac{w_0}{3}\left\{b_3 - \frac{3}{2}\frac{b_2}{w_0} - p^3 - \frac{1}{p^3} + \frac{1}{2w_0^3} + \frac{3}{2w_0}\left(p^2 + \frac{1}{p^2}\right)\right\}$$

where b_2 and b_3 are coefficients of a function in $\mathscr{P}(b_1)$.

3. **Arc length.** For $f(z) \in U(p)$ the arc length of the image of the circle $|z|=r$ under the mapping $w=f(z)$ is $L_r(f)=\int_{|z|=r}|f'(z)|\,|dz|$. We now prove the following.

THEOREM 2. *If $f(z)\in U^*(p, w_0)$, then*

$$L_r(f) = O\left(\log\frac{1}{|r-p|\,(1-r)}\right).$$

PROOF. Suppose $f(z)\in U^*(p, w_0)$, and let $M_r=\max_{|z|=r}|f(z)-w_0|$. The number M_r is finite for $r\neq p$ and $r<1$. Further, M_r is bounded near $r=1$. Then if $r\neq p$, we have

$$\begin{aligned} L_r(f) &= \int_0^{2\pi} |f'(z)|\, r\,d\theta = \int_0^{2\pi} \left| z\frac{f'(z)}{f(z)-w_0} \right| |f(z)-w_0|\, d\theta \\ &\leqq M_r \int_0^{2\pi} \left| z\frac{f'(z)}{f(z)-w_0} + \frac{p}{z-p} - \frac{pz}{1-pz} \right| d\theta \\ &\quad + M_r \int_0^{2\pi} \left| \frac{p}{z-p} - \frac{pz}{1-pz} \right| d\theta \\ &= M_r \int_0^{2\pi} |P(z)|\, d\theta + M_r \int_0^{2\pi} \left| \frac{p(1-z^2)}{(z-p)(1-pz)} \right| d\theta, \end{aligned}$$

where $\operatorname{Re}\{P(z)\}\geqq 0$. Since $P(z)$ is subordinate to $(1+z)/(1-z)$, we have

$$\begin{aligned} L_r(f) &\leqq M_r \int_0^{2\pi} \left| \frac{1+z}{1-z} \right| d\theta + M_r \frac{p+pr}{1-pr} \int_0^{2\pi} \left| \frac{1-z}{p-z} \right| d\theta \\ &\leqq 2\pi M_r + 4M_r \log\frac{1+r}{1-r} \\ &\quad + 2\pi M_r \frac{(1+r)(p+r^2)}{(1-pr)(p^2+r^2)} + 4M_r \log\left(\frac{p+r}{|p-r|}\right). \end{aligned}$$

Hence $L_r(f)=O(\log(|r-p|(1-r))^{-1})$.

For bounded regular univalent starlike functions Keogh [3] has shown that $L_r(f)=O(\log(1-r)^{-1})$. Hayman [1] has shown that O may not be replaced by o. For our case the function $f(z)$ is bounded near $|z|=1$ but has a pole at $z=p$. Thus $L_r(f)$ depends on both $(1-r)$ and $|r-p|$. Further, Hayman's example shows that, in Theorem 2, O may not be replaced by o for $0<p<2\sqrt{3}$.

4. **An integral representation and a bound on the modules for $U^*(p, w_0)$.** The following integral representation for functions in $U^*(p, w_0)$ may be obtained from equation (1).

THEOREM 3. *If $f(z)\in U^*(p, w_0)$, then*

$$\lim_{r\to 1} \arg(f(re^{i\theta})-w_0)$$

exists for all θ, and

$$\frac{f(z)-w_0}{w_0} = \frac{p}{(z-p)(1-pz)} \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1-e^{it}z)\, dV(t)\right\}, \tag{7}$$

where

$$V(t) = -\lim_{r\to 1} \arg((f(re^{it})-w_0)/w_0),$$

and

$$\frac{1}{\pi}\int_0^{2\pi} e^{-it}\,dV(t) = p + 1/p + 1/w_0. \tag{8}$$

PROOF. We note from (1) that for $f(z)\in U(p, w_0)$ there is a corresponding regular function $P(z)$ with $P(0)=1$, $\operatorname{Re}\{P(z)\}\geqq 0$, and $P'(0)=p+1/p+1/w_0$. The function $P(z)$ has the Herglotz integral representation

$$P(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}\,dF(t),$$

where $F(t)$ is nondecreasing over $[0, 2\pi]$, $F(2\pi)-F(0)=2\pi$, $F(t)-t$ is periodic with period 2π, and $(2\pi)^{-1}\int_0^{2\pi} dF(t) = 1$. Without loss of generality we may assume that $F(t)$ is normalized so that $F(t)=\frac{1}{2}[F(t-0)+F(t+0)]$. Equation (1) may now be written as

$$\begin{aligned} z\frac{f'(z)}{f(z)-w_0} &= \frac{pz}{1-pz} - \frac{p}{z-p} - \frac{1}{2\pi}\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}\,dF(t) \\ &= \frac{pz}{1-pz} - \frac{z}{z-p} - \frac{1}{2\pi}\int_0^{2\pi}\frac{2e^{-it}z}{1-e^{-it}z}\,dF(t). \end{aligned}$$

Dividing by z and integrating from 0 to z we obtain

$$\log\left|\frac{f(z)-w_0}{w_0}\right| = \log\frac{p}{(z-p)(1-pz)} + \frac{1}{\pi}\int_0^{2\pi}\log(1-e^{-it}z)\,dF(t).$$

Integrating by parts, and observing that $\int_0^{2\pi}\log(1-e^{-it}z)\,dt=0$, we have

$$\begin{aligned} \log\left|\frac{f(z)-w_0}{w_0}\right| &= \log\frac{p}{(z-p)(1-pz)} - \frac{i}{\pi}\int_0^{2\pi}\frac{e^{-it}z}{1-e^{-it}z}(F(t)-t)\,dt \\ &= \log\frac{p}{(z-p)(1-pz)} - \frac{i}{2\pi}\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}(F(t)-t)\,dt, \end{aligned}$$

since $\int_0^{2\pi}(F(t)-t)\,dt=0$. By taking imaginary parts we find that

$$\begin{aligned} \arg\left|\frac{f(z)-w_0}{w_0}\right| &= -\theta + \arg\left(\frac{pz}{(z-p)(1-pz)}\right) \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}(F(t)-t)\,dt. \end{aligned} \tag{9}$$

By Fejer's Theorem, (9) gives

$$\lim_{r\to 1}\arg\left((f(re^{i\theta})-w_0)/w_0\right) = -\theta - F(\theta) + \theta$$

for a certain branch of the argument. Thus $\lim_{r\to 1}[f(re^{i\theta})-w_0]$ exists for all θ and $V(t)=F(t)+C$ for some constant C. Moreover, since $\arg[f(re^{it})-w_0]$ is a decreasing function of t on $[0, 2\pi]$, for r near 1 we have that $V(t)$ is a nondecreasing function on $[0, 2\pi]$.

Since the function $V(t)$ is nondecreasing over $[0, 2\pi]$, there is a point t_0 in $[0, 2\pi]$ such that at t_0 the function $V(t)$ has a maximum jump. We now prove the following result.

THEOREM 4. *Suppose $f(z)\in U^*(p, w_0)$ and define*

$$V(t) = \lim_{r\to 1} \arg (w_0/(f(re^{it}) - w_0)).$$

Suppose α is such that $\alpha\pi$ is the maximum jump of $V(t)$. Then

$$\left|\frac{f(z) - w_0}{w_0}\right| \leqq \frac{2^{2-\alpha}p(1 + r)^\alpha}{|r - p|\,(1 - pr)}. \tag{10}$$

PROOF. From (7) we have that

$$\begin{aligned}\log\left|\frac{f(z) - w_0}{w_0}\right| &= \log\frac{p}{|z - p|\,|1 - pz|}\\ &\quad + \frac{1}{\pi}\int_0^{2\pi} \log|1 - e^{-it}z|\,dV(t).\end{aligned} \tag{11}$$

Since

$$\log\left|\frac{1 - e^{-it}z}{2}\right| < 0$$

we have that

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} \log|1 - e^{-it}z|\,dV(t) &= \frac{1}{\pi}\int_0^{2\pi}\log\left|\frac{1 - e^{-it}z}{z}\right| dV(t) + 2\log 2\\ &\leqq \frac{\alpha\pi}{\pi}\log\left|\frac{1 - e^{-it_0}z}{2}\right| + 2\log 2\\ &\leqq \log[2^{2-\alpha}(1 + r)^\alpha].\end{aligned} \tag{12}$$

Together (11) and (12) yield

$$\left|\frac{f(z) - w_0}{w_0}\right| \leqq \frac{2^{2-\alpha}p(1 + r)^\alpha}{|z - p|\,|1 - pz|} \leqq \frac{2^{2-\alpha}p(1 + r)^\alpha}{|r - p|\,(1 - pr)}.$$

REMARK. For $w_0=-p/(1+p)^2$ and for all z, $p<|z|<1$, inequality (10) becomes

$$|f(z)| \leqq r\Big/\left(-1 + \left(p + \frac{1}{p}\right)r - r^2\right),$$

which is the bound obtained by Komatu **[4]** for functions in $U(p)$ with $|a_2|>2$. Further, for $w_0=p/(1+p)^2$, inequality (7) becomes

$$|f(z)| \leqq r/(1-r)^2$$

as $p\to 1$, which is the well-known bound for regular univalent functions.

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