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# PÓLYA'S PROPERTY W AND FACTORIZATION— A SHORT PROOF

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ABSTRACT. For an $n$th order linear differential expression, the equivalence of Pólya's Property W and factorization into first order expressions is proven directly and briefly.

The usual proof that a linear differential expression with Pólya's Property W admits a factorization into first order expressions employs Jacobi's formula for the minors of the adjugate matrix [1]. (For Jacobi's theorem, see [2].) A simple, direct proof is given here based on the following two elementary lemmas.

LEMMA 1. *If $I$ is an open interval, $f \in C^1(I)$, $f$ is nonzero on $I$, and*

$$J(y) = f(d/dx)(yf^{-1})$$

*then $J(f)=0$ and $J$ is a first order linear differential expression with leading coefficient equal to one.*

PROOF. The lemma is verified by an easy computation.

NOTATION. If $\{h_k\}_{k=1}^m \subset C^{m-1}(I)$, we let $W(h_1, \cdots, h_m)$ denote the Wronskian determinant of this set of functions.

LEMMA 2. *If $\{h_k\}_{k=1}^m \subset C^m(I)$, $W(h_1, \cdots, h_m)(x) \neq 0$, $\forall x \in I$, and*

$$K(y) = W(h_1, h_2, \cdots, h_m, y)[W(h_1, h_2, \cdots, h_m)]^{-1}$$

*then $K$ is the unique $m$th order linear differential expression with leading coefficient equal to one for which $\{h_k\}_{k=1}^m$ is a fundamental set.*

PROOF. Expansion of the determinant in terms of $y^{(k)}$, $k=0, \cdots, m$, and the corresponding cofactors verifies the form of $K(y)$. If $K_1$ were a second such expression, then $K(y)-K_1(y)$ would be of order $(m-1)$ with $m$ linearly independent solutions.

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THEOREM. *Let $L(y)$ be a linear differential expression of order $n$ with continuous coefficients and with leading coefficient equal to one. Let $\{h_k\}_{k=1}^m$ be a fundamental set for $L$ such that $W_m(x)=W(h_1,\cdots,h_m)(x)\neq 0$, $\forall x\in I$, $m=1,\cdots,n-1$. Then*

$$L(y)=\frac{W_n}{W_{n-1}}\frac{d}{dx}\frac{W_{n-1}^2}{W_nW_{n-2}}\cdots\frac{d}{dx}\frac{W_2^2}{W_3W_1}\frac{d}{dx}\frac{W_1^2}{W_2}\frac{d}{dx}\frac{y}{W_1}.$$

PROOF. Set $W_0=1$. Set

$$L_k(y)=\frac{W_k}{W_{k-1}}\frac{d}{dx}\left(\frac{W_{k-1}}{W_k}y\right),\qquad k=1,\cdots,n.$$

Then $L_1(h_1)=0$, and $L_1$ has leading coefficient equal to one. Inductively, we assume $L_k(L_{k-1}(\cdots L_2(L_1(y))\cdots))=(\prod_{i=1}^k L_i)(y)$ maps $h_m$ to zero for $m=1,\cdots,k$ and has leading coefficient equal to one. Then

$$\left(\prod_{i=1}^k L_i\right)(h_{k+1})=W_{k+1}W_k^{-1}$$

by Lemma 2. So $L_{k+1}(\prod_{i=1}^k L_i(h_{k+1}))=0$ by Lemma 1. The composition of two expressions with leading coefficient equal to one is again of that form. Thus $\prod_{i=1}^n L_i$ has $\{h_k\}_{k=1}^n$ as a fundamental set. Hence, by Lemma 2, $\prod_{i=1}^n L_i=L$.

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