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SEPARATING p-BASES AND TRANSCENDENTAL EXTENSION FIELDS

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ABSTRACT. Let L/K denote an extension field of characteristic $p \neq 0$. It is known that if L/K has a finite separating transcendence base, then every relative p-base of L/K is a separating transcendence base of L/K. In this paper we show that when every relative p-base of L/K is a separating transcendence base of L/K, then the transcendence degree of L/K is finite. We also illustrate the connection between the finiteness of transcendence degree of L/K and the property that $L/K(X)$ is separable algebraic for every relative p-base X of L/K.

Let L/K denote an extension field of characteristic $p>0$. If X is a relative p-base such that $L/K(X)$ is separable algebraic, then we call X a separating relative p-base. When every relative p-base of L/K is a separating relative p-base we say that L/K is of type R_s. Let S denote the set of all intermediate fields of L/K. When every element of S is of type R_s (with respect to K), we say that L/K is of type $R_s(S)$. This notation extends that used by the authors in **[4]** where a purely inseparable extension L/K is called type R when $L=K(X)$ for every relative p-base X, and where it is shown that L/K is of type $R(S)$ if and only if L/K has an exponent.

In this paper we give four theorems that illustrate the connection between type R_s and the finiteness of transcendence degree. We make use of relevant results that appear in Mac Lane **[3]** and Dieudonné **[1]**.

Finitely generated extensions, whose measures of inseparability have recently been analyzed anew by Kraft **[2]**, are a subset of extensions of type $R_s(S)$, a fact that follows easily from Theorem 2 below.

LEMMA. *L/K is of type R_s if and only if there is no intermediate field L' of L/K such that $L=L'(L^p)$ and L/L' is not separable algebraic.*

PROOF. Suppose L/K is not of type R_s. Then there exists a relative p-base X which is not a separating relative p-base. Hence if we set $L'=K(X)$, then $L=L'(L^p)$ and L/L' is not separable algebraic. On the other hand, suppose there exists an intermediate field L' of L/K such that

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$L=L'(L^p)$ and L/L' is not separable algebraic. Then L' contains a relative p-base of L/K which is not a separating relative p-base. Q.E.D.

COROLLARY. *If L/K is of type R_s, then L/L' is of type R_s for every intermediate field L' of L/K.*

PROOF. Suppose L/K is of type R_s and that L/L' is not of type R_s for some intermediate field L' of L/K. Then there exists an intermediate field L'' of L/L' such that $L=L''(L^p)$ and L/L'' is not separable algebraic. This is a contradiction because L'' is also an intermediate field of L/K. Q.E.D.

We call L/K separable when the tensor product $L\otimes_K K^{p^{-1}}$ is a field.

THEOREM 1. *When L/K is separable, the following statements are equivalent.*

(1) *L/K is of type $R_s(S)$.*

(2) *L/K is of type R_s.*

(3) *L/K has a finite separating transcendence base.*

(4) *Every relative p-base of L/K is a separating transcendence base of L/K.*

(5) *Every relative p-base of L/K is a transcendence base of L/K.*

(6) *The transcendence degree of L/K equals the imperfection degree of L/K, and these are finite.*

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). That (1) implies (2) is immediate. Suppose (2) holds. Then, by [**3**, Theorem 11, p. 381], L/K has a separating transcendence base T. Suppose T is infinite. Let $T_0=\{t_1, t_2, \cdots\}$ be a denumerable subset of T and set $T'=T-T_0$ (set difference), $K'=K(T')$. Then T_0 is a separating transcendence base of L/K'. T_0 is therefore a relative p-base of L/K', hence the set $T_0'=\{t_1t_2^p, t_2t_3^p, \cdots\}$ is a relative p-base of L/K'. By our corollary, $L/K'(T_0')$ is separable algebraic. Hence $t_1\in K'(T_0', t_1^p)$. However this contradicts the algebraic independence of T_0 over K'. Thus T is finite.

(3)$\Rightarrow$(4). This implication follows from [**3**, Corollary, p. 385].

(4)$\Rightarrow$(5). This follows from [**3**, Theorem 13, p. 383].

(5)$\Rightarrow$(6). That the transcendence degree of L/K equals the imperfection degree of L/K is immediate. The finiteness condition follows from the equivalence of (4) and (5) and the proof of (2) implies (3).

(6)$\Rightarrow$(1). By [**3**, Theorem 11, p. 381], we have (3). Hence an application of [**3**, Theorem 17, p. 386] and [**3**, Corollary, p. 385] yields (1).

Q.E.D.

When there exists an integer $e\geqq 0$ such that $K(L^{p^e})/K$ is separable but $K(L^{p^{e-1}})/K$ is not, then e is called the inseparability exponent of L/K, as in [**2**, p. 111]. When L/K has an inseparability exponent, there exist certain maximal separable intermediate fields of L/K whose construction (for our case) is indicated by Dieudonné [**1**, p. 17] (see also [**3**, p. 384]) as follows: From a relative p-base X of L/K select a subset Y such that Y^{p^e} is a relative p-base of $K(L^{p^e})/K$. Since the latter extension is separable, Y is algebraically independent over K, and since $K(L^{p^e})/K(Y^{p^e})$ is separable, so is $K(L^{p^e}, Y)/K(Y)$. Set $F=K(L^{p^e}, Y)$. Then F/K is a maximal separable intermediate field of L/K and L/K is isomorphic over F to a subfield of the field $F\otimes_K K^{p^{-\infty}}$. Such an intermediate field Dieudonné has called distinguished maximal separable. When L/K has a finite relative p-base, the degree of L over any distinguished maximal separable intermediate field is Weil's order of inseparability of L/K ([**1**, pp. 14, 17], [**2**, p. 111]). The use of the term "distinguished" is consistent with that used by the authors in [**5**]. This follows from application of [**5**, Proposition 1.10, p. 5] to the fact that Y is a relative p-base of F/K and relatively p-independent in L/K. In the finitely generated case, a distinguished maximal separable intermediate field F is the same as the optimal separable intermediate field denoted by K_0 in [**2**, p. 111]. In fact, $K(F^{p^e})=K(L^{p^e})$ holds in our more general context.

For a transcendence base T of L/K, let S_T denote the maximal separable intermediate field of $L/K(T)$.

THEOREM 2. *When L/K is arbitrary, the following statements are equivalent.*

(1) *L/K is of type $R_s(S)$.*

(2) *L/K has finite transcendence degree and L/S_T has an exponent for every transcendence base T of L/K.*

(3) *L/K has finite transcendence degree and L/S_T has an exponent for some transcendence base T of L/K.*

(4) *L/K has an inseparability exponent e and $K(L^{p^e})/K$ has a finite separating transcendence base.*

(5) *L/K has a distinguished maximal separable intermediate field of type R_s.*

(6) *L/K has a distinguished maximal separable intermediate field and every such field is of type R_s.*

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). If T is any transcendence base of L/K, then (1) implies that L'/K is of type R_s for every intermediate field L' of the purely inseparable extension L/S_T. Since by our corollary L'/S_T is also of type R_s, L'/S_T is actually of type R. Thus L/S_T is of type $R(S)$, hence L/S_T has an exponent by [**4**, Corollary, p. 240]. Since S_T/K is also of type R_s, T is finite by Theorem 1 above. Thus (1) implies (2). That (2) implies (3) is trivial.

(3)$\Rightarrow$(1). If L' is an intermediate field of L/K, then a transcendence base Z' of L'/K can be extended to a transcendence base Z of L/K. Let T

be the transcendence base of L/K satisfying (3). Since T is finite, there exists a positive integer m such that $T^{p^m} \subseteq S_Z$. Hence L/S_Z has an exponent. Now $S_Z \supseteq S_{Z'}$, $S_Z/K(Z')$ is separable and $S_{Z'}/K(Z')$ is, in particular, relatively perfect. Hence $S_Z/S_{Z'}$ is separable by [**1**, Proposition 6, p. 8]. Thus $L' \cap S_Z = S_{Z'}$. Since L/S_Z has an exponent, say n, $L'^{p^n} \subseteq L' \cap S_Z = S_{Z'}$. Thus property (3) is inherited by every intermediate field of L/K. Hence it suffices to show that (3) implies L/K is of type R_s. Now T is finite, so S_T/K is of type $R_s(S)$ by Theorem 1. Since $L^{p^e} \subseteq S_T$ for some integer $e \geqq 0$, we have that $K(L^{p^e})/K$ is of type R_s. If X is any relative p-base of L/K, X^{p^e} contains a relative p-base of $K(L^{p^e})/K$. Hence $K(L^{p^e})/K(X^{p^e})$ is separable algebraic, whence $K(L^{p^e}, X)/K(X)$ is separable algebraic. Since $L = K(L^{p^e}, X)$ and X was arbitrary, we have that L/K is of type R_s.

(3)$\Leftrightarrow$(4). That (4) implies (3) follows easily. To show that (3) implies (4), note that by (3), L/K has an inseparability exponent, say e. Since (3) and (1) are equivalent, $K(L^{p^e})/K$ is of type R_s. Hence, by Theorem 1, $K(L^{p^e})/K$ has a finite separating transcendence base.

(4)$\Leftrightarrow$(5). To show (5) implies (4), let $F = K(L^{p^e}, Y)$ be a distinguished maximal separable intermediate field such that F/K is of type R_s. Since F/K is separable, every relative p-base of F/K is a finite separating transcendence base. Hence Y is a finite separating transcendence base of F/K. Thus Y^{p^e} is a finite separating transcendence base of $K(L^{p^e})/K$. To show that (4) implies (5), note that $L/K(L^{p^e})$ has an exponent and $K(L^{p^e})/K$ has a finite separating transcendence base. By (3) implies (1), L/K is of type $R_s(S)$. Hence F/K is of type R_s.

(4)$\Leftrightarrow$(6). (6) implies (5) trivially and we have proved (5) implies (4). Hence (6) implies (4). Assume (4). Now, all distinguished maximal separable intermediate fields contain $K(L^{p^e})/K$. Hence the proof that (4) implies (5) applies to every such distinguished intermediate field. Q.E.D.

THEOREM 3. *When L/K has finite transcendence degree, the following statements are equivalent.*

(1) *L/K is of type R_s.*

(2) *L/S_T is of type R for every transcendence base T of L/K.*

(3) *L/S_T is of type R for some transcendence base T of L/K.*

PROOF. That (1) implies (2) follows from the Corollary. That (2) implies (3) is trivial. To show that (3) implies (1), let X be a relative p-base of L/K. Then X contains a relative p-base of L/S_T. Since L/S_T is of type R, $L = S_T(X) \supseteq K(T, X)$. Thus $L/K(T, X)$ is separable algebraic and by hypothesis T is finite. If $L/K(X)$ is not separable algebraic, then $K(T, X)/K(X)$ is not separable algebraic. Then $K(T, X)/K(X)$ has a nonempty relative p-base, because $K(T, X)/K(X)$ is finitely generated. But since $L/K(T, X)$ is separable algebraic, $L/K(X)$ has a nonempty relative p-base, contrary to the fact that $L/K(X)$ is relatively perfect. Q.E.D.

When an extension field has a separating transcendence base, we say it is separably generated.

THEOREM 4. *When L/K contains a separable intermediate field F/K such that L/F is finite degree purely inseparable, the following statements are equivalent.*

(1) *L/K is of type $R_s(S)$.*

(2) *L/K is of type R_s.*

(3) *The transcendence degree of L/K is finite and F/K is separably generated.*

(4) *The imperfection degree of L/K is finite and F/K is separably generated.*

(5) *F/K is of type R_s.*

(6) *$K(L^{p^e})/K$ is of type R_s for some integer $e \geqq 0$.*

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3). That (1) implies (2) is immediate. Suppose (2) holds. Let X be a relative p-base of L/F. Then $X^{p^e} \subseteq F$ for some integer $e \geqq 0$ and X is finite. Let Y be a relative p-base of F/K. Since L/K is of type R_s and $X \cup Y$ contains a relative p-base of L/K, $L/K(X, Y)$ is separable algebraic. Hence $K(L^{p^e}, Y)/K(X^{p^e}, Y)$ is separable algebraic. Now $K(L^{p^e}, Y) \subseteq F = K(F^{p^e}, Y) \subseteq K(L^{p^e}, Y)$. Thus $F = K(L^{p^e}, Y)$, so $F/K(X^{p^e}, Y)$ is separable algebraic. Since F/K is separable, $F/K(Y)$ is separable. Hence $K(Y, X^{p^e})/K(Y)$ is separable and finitely generated. If the latter extension has a nonempty relative p-base, then we contradict the fact that $F/K(Y)$ is relatively perfect and $F/K(Y, X^{p^e})$ is separable algebraic. Hence $F/K(Y)$ is separable algebraic, so F/K is of type R_s. Since F/K is also separable, it has a finite separating transcendence base by Theorem 1. Since L/F is algebraic, L/K has a finite transcendence base.

(3)$\Rightarrow$(4)$\Rightarrow$(5). That (3) implies (4) is routine. Suppose (4) holds. Then the transcendence degree of F/K must be finite. Hence by Theorem 1, F/K is of type R_s.

(5)$\Rightarrow$(6). Let e be a positive integer such that $L^{p^e} \subseteq F$. Since F/K is of type R_s, F/K is of type $R_s(S)$. Hence $K(L^{p^e})/K$ is of type R_s.

(6)$\Rightarrow$(1). Since $K(L^{p^e})/K$ is of type R_s, $K(L^{p^e})/K(X^{p^e})$ is separable algebraic for every relative p-base X of L/K. Hence $K(L^{p^e}, X)/K(X)$ is separable algebraic, that is L/K is of type R_s. Hence, as in the proof of (2) implies (3), L/K has finite transcendence degree and F/K is of type R_s. Thus, replacing S_T by F in Theorem 2, we have that (1) holds. Q.E.D.

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