DETERMINATION OF k_n ($n \geq 3$) FOR GLOBAL FIELDS¹

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ABSTRACT. A short proof is obtained for Tate's theorem about k_n of global fields.

For a field F , the groups $K_n F$ and $k_n F = K_n F / 2K_n F$ are first defined by Milnor [1]. Following the notations there, we shall write $K_1 F = \{l(a) : a \neq 0\}$ where $l(ab) = l(a) + l(b)$. In this note, we present a simple proof of the following theorem of Tate (see Appendix of [1]):

THEOREM. *If F is a global field, and $\{F_v\}$ is the family of real completions of F , then for $n \geq 3$, $k_n F \rightarrow \bigoplus_v k_n F_v$ is an isomorphism. In particular, the dimension of $k_n F$ as a $\mathbb{Z}_2$ -vector space is equal to the number of real completions of F .²*

We shall need two lemmas.

LEMMA 1. *If x, y and $z = x + y$ are nonzero in F , then $l(x)l(y) = l(z)l(-xy)$ in $k_2 F$.*

PROOF. Since $xz^{-1} + yz^{-1} = 1$, we have $l(xz^{-1})l(yz^{-1}) = 0$ by definition of $K_2(F)$. Therefore in $k_2 F$,

$$(l(x) + l(z))(l(y) + l(z)) = 0.$$

This implies

$$\begin{aligned} l(x)l(y) &= l(z)l(z) + l(z)l(xy) \\ &= l(z)l(-1) + l(z)l(xy) \\ &= l(z)l(-xy) \quad \text{in } k_2 F. \end{aligned}$$

REMARK. The same proof shows that $l(x)l(y) = l(z)l(-yx^{-1})$ in $K_3 F$.

LEMMA 2. *For nonzero a, b, c in F , $l(a)l(b)l(c) = 0$ in $k_3 F$ unless there exists a real completion of F at which a, b, c are all negative.*

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² It can thus be shown that $k_n F$ ($n \geq 3$) is isomorphic to the multiplicative group of F modulo sums of (four) squares.

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Note that this is a refinement of Lemma A5 in [1]. Therefore, as noted in [1], once we prove the above lemma, the theorem will follow as an immediate consequence. In [1] Tate's proof of Lemma 2 (or rather its weaker version, Lemma A5 in [1]) invokes rather deep properties involving idèle groups, Kummer extensions and reciprocity laws. Our present proof, however, is based on a straightforward application of the Hasse-Minkowski theorem.

PROOF OF LEMMA 2. Suppose, in every real completion F_v , at least one of a, b, c is positive. We claim that the quadratic form $q = aX^2 + bY^2 - abW^2 + cT^2$ represents 1 in F . By the Hasse-Minkowski theorem, it suffices to verify that q represents 1 at all completions of F . At any real completion F_v , this is obvious because at least one of a, b, c is positive in F_v . At all other completions (finite primes and 'complex' primes), q actually represents all nonzero elements, since any four-dimensional quadratic form is universal over such completions [2, 63:19]. Therefore, there exist x, y, w, t in F such that $ax^2 + by^2 - abw^2 + ct^2 = 1$. If x, y, w, t are all nonzero, and also $ax^2 + by^2 \neq 0$, then in k_3F ,

$$\begin{aligned} l(a)l(b)l(c) &= l(ax^2)l(by^2)l(c) \\ &= l(ax^2 + by^2)l(-ab)l(c) && \text{(by Lemma 1)} \\ &= l(ax^2 + by^2)l(-abw^2)l(ct^2) \end{aligned}$$

and this is zero in k_3F according to Lemma 1.3 in [1]. If $ax^2 + by^2 = 0$, or if some of x, y, w, t are zero, we observe that

$$l(a)l(b)l(c) = l(-ab)l(b)l(c) = l(a)l(-ab)l(c),$$

and a similar argument also shows $l(a)l(b)l(c) = 0$ in k_3F . This concludes the proof.

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