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# RIGID PAIRS OF LONG ARCS[1]

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ABSTRACT. Arcs (linearly ordered continua) $A$ and $B$ are constructed such that every map from $A$ to $B$ and every map from $B$ to $A$ is constant. The Generalized Continuum Hypothesis is sufficient for the existence of two such arcs each of cardinality $2^{\aleph}$ for each uncountable cardinal $\aleph$.

1. **Introduction.** By an *arc* we mean a compact connected Hausdorff space with exactly two noncut points. A separable arc is then a homeomorph of the real interval $I=[0, 1]$. An arc is *long* if it is not separable or equivalently not metrizable. A survey of the literature on long arcs is given in [**2**].

Here we construct arcs $A$ and $B$ such that if $f:A\to B$ and $g:B\to A$ are continuous, then the image of each function is a point. We say such a pair of arcs is *rigid*. Due to Urysohn's lemma, neither $A$ nor $B$ contains a separable arc. Moreover if $C$ and $D$ are subarcs of a rigid pair of arcs $A$ and $B$ respectively, then $C$ and $D$ form a rigid pair. This follows immediately since subarcs are retracts. The desired arcs are obtained by constructing two kinds of simply ordered sets.

Let $X$ be a simply ordered set (chain) with at least two elements. The *intrinsic topology* of $X$ is obtained by taking open intervals and open rays as a basis. $X$ is then an arc if $X$ has a supremum, every nonempty subset of $X$ has an infimum, and for any two points $x<y$ there is a point $z$, $x<z<y$ [**1**, X, §§7–8].

2. **Arcs from sequences of ordinals.** The usual construction of the unit interval from sequences of digits involves the identification of discontinuities, e.g., $0.4999\cdots$ and $0.5000\cdots$. In this section, long arcs are constructed in somewhat the same fashion from sequences of ordinals. Here identification is unnecessary.

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Let $\omega$ be the first infinite ordinal and let $\lambda$ be any limit ordinal. Then $\lambda^\omega$ denotes the set of all sequences of ordinals less than $\lambda$. Order $\lambda^\omega$ lexicographically, i.e., $x(0), x(1), \cdots < y(0), y(1), \cdots$ if and only if there is an $n$ such that $x(n) < y(n)$ and $x(i) = y(i)$ for all $i < n$. A discussion of such ordinal powers may be found in [**1**]. The simply ordered set obtained by adjoining a supremum 1 to $\lambda^\omega$ is denoted $\lambda^\omega + 1$.

THEOREM 1. *$\lambda^\omega + 1$ is an arc.*

PROOF. Suppose $M$ is a nonempty subset of $\lambda^\omega$. Let $m(0)$ be the infimum of the set of first terms of sequences in $M$. Assume further $m(i)$ is defined for all $i < n$. Then let $m(n)$ be the infinimum of the set of $n$th terms of sequences in $M$ whose first $n-1$ terms are $m(0), m(1), \cdots, m(n-1)$. The sequence $m(0), m(1), \cdots, m(n), \cdots$ is the infimum of $M$ in $\lambda^\omega$. Therefore every nonempty subset of $\lambda^\omega + 1$ has an infimum. Now suppose $x, y \in \lambda^\omega + 1$ and $x < y$. Assume $y = 1$. Since $\lambda$ is a limit ordinal, $x(0)$ has successor $x(0) + 1 < \lambda$. Let $z$ be the sequence $x(0)+1, 0, 0, 0, \cdots$. Then $x < z < y$. Finally assume $y \in \lambda^\omega$. Then there is an $n$ such that $x(n) < y(n)$ and $x(i) = y(i)$ for all $i < n$. Let $z = x(0), x(1), \cdots, x(n), x(n+1)+1, 0, 0, 0, \cdots$. Then $x < z < y$. This completes the proof.

COROLLARY. *The nonnegative reals are isomorphic to the set of all sequences of nonnegative integers ordered lexicographically.*

THEOREM 2. *$\lambda^\omega$ contains a subset $D$ such that each point of the space is the limit of a descending sequence of points of $D$.*

PROOF. Let $D$ be the set of all $x$ such that $x$ is eventually the ordinal 0, i.e., such that there is an $n < \omega$ and $x(m) = 0$ for all $m$, $n < m < \omega$. Suppose $x \in \lambda^\omega$. For each $n < \omega$, let $d_n$ be the following point of $D$. $d_n(m) = x(m)$ for all $m < n$, $d_n(n)$ is the successor of $x(n)$, and $d_n(m) = 0$ for all $m > n$. Then $d_n > d_{n+1} > x$ and $\text{limit}\{d_n\} = x$.

If $X$ is a set, we denote the cardinal of $X$ by $\aleph(X)$.

THEOREM 3. *Every subarc of $\lambda^\omega$ contains $\aleph(\lambda)$ pairwise disjoint open sets.*

PROOF. Let $x, y \in \lambda^\omega + 1$ such that $x < y$. Then there is an $n < \omega$ such that $x(n) < y(n)$ and $x(i) = y(i)$ for all $i < n$. For each $\alpha \in \lambda$, let $U_\alpha$ be the open interval $(\alpha_1, \alpha_2)$ where $\alpha_1(i) = x(i)$ for all $i < n$, $\alpha_1(n+1) = x(n+1)+1$, $\alpha_1(n+2) = \alpha$, $\alpha_1(i) = 0$ for all $i > n+3$, $\alpha_1(i) = \alpha_2(i)$ for all $i \neq n+3$, $\alpha_1(n+3) = 0$ and $\alpha_2(n+3) = 1$. Then $\{U_\alpha\}$ is a collection of pairwise disjoint open intervals each in $(x, y)$, the cardinal of this collection is $\aleph(\lambda)$.

3. **Ordinal powers of the unit interval.** V. Novák [**4**] has shown that $I^\mu$ is an arc for any ordinal $\mu$. Here $I^\mu$ is the set of all functions from $\mu$ to $I$ ordered lexicographically.

Let $\mu$ be any limit ordinal such that for all $\alpha<\mu$, $\aleph(\alpha)<\aleph(\mu)$.

THEOREM 4. *$I^\mu$ contains a dense subset $M$ such that every subset with a cluster point in $M$ has cardinality at least $\aleph(\mu)$.*

PROOF. Let $M=\{x$: there is an $\alpha<\mu$ such that $0<x(\beta)<1$ for all $\beta$, $\alpha<\beta<\mu\}$. Suppose $a, b\in I^\mu$ and $a<b$. Let $\alpha$ be the first ordinal such that $a(\alpha)<b(\alpha)$. Let $C\in I$ such that $a(\alpha)<C<b(\alpha)$. Define $x$ by (1) $x(\beta)=a(\beta)$ for all $\beta<\alpha$, and (2) $x(\beta)=C$ for all $\beta$, $\alpha<\beta<\mu$. Then $x\in M$ and $a<x<b$. Therefore $M$ is a dense subset of $I^\mu$.

Now suppose $X$ is a subset of $I^\mu$, $x\in M\cap X$ and each neighborhood of $x$ contains a point of $X$ distinct from $x$. Index $X-x$ so that points are not indexed twice, $X-x=\{x_\gamma:\gamma<\zeta\}$. Define $\psi:\zeta\to\mu$ as follows. For each $\gamma<\zeta$, let $\psi(\gamma)$ be the first $\eta<\mu$ such that $x(\eta)\neq x_\gamma(\eta)$. Suppose there is a $\delta<\mu$ such that $\psi(\gamma)<\delta$ for all $\gamma<\zeta$. Let $\alpha<\mu$ be an ordinal such that $0<x(\beta)<1$ for all $\beta$, $\alpha<\beta<\mu$. Let $a, b\in I^\mu$ such that $a(\gamma)=b(\gamma)=x(\gamma)$ for all $\gamma<\max[\alpha, \delta]$ and $a(\max[\alpha, \delta])<x(\max[\alpha, \delta])<b(\max[\alpha, \delta])$. Then the interval $(a, b)$ is a neighborhood of $x$ which contains no point of $X$ distinct from $x$. This involves a contradiction. Thus image $\psi$ is cofinal with $\mu$. Therefore $\aleph(\zeta)\geqq\aleph(\mu)$. This completes the proof.

THEOREM 5. *$I^\mu$ contains a dense subset with cardinality $\sum_{\alpha<\mu} 2^{\aleph_0\cdot\aleph(\mu)}$.*

PROOF. Let $D$ be the set of all points which are eventually 0. Then $D$ is a dense subspace of $I^\mu$.

$$\aleph(D) = \aleph\left(\bigcup_{\alpha<\mu} I^\alpha\right) = \sum_{\alpha<\mu} \aleph(I^\alpha) = \sum_{\alpha<\mu} 2^{\aleph_0\cdot\aleph(\alpha)}.$$

4. **The main result.** Let $\mu$ and $\lambda$ be any limit ordinals such that $\mu$ is uncountable and $\aleph(\lambda)>\sum_{\alpha<\mu} 2^{\aleph(\alpha)}$. If we assume the Continuum Hypothesis, we could take $\mu=\Omega$, the first uncountable ordinal, and $\lambda$ the first ordinal such that $\aleph(\lambda)>2^{\aleph_0}$.

THEOREM 6. *The arcs $I^\mu$ and $\lambda^\omega+1$ form a rigid pair.*

PROOF. Suppose $f$ is a nonconstant map from $I^\mu$ to $\lambda^\omega+1$. By Theorem 5, $I^\mu$ contains a dense subset $D$ with $\aleph(D)=\sum_{\alpha<\mu} 2^{\aleph_0\cdot\aleph(\alpha)}=\sum_{\alpha<\mu} 2^{\aleph(\alpha)}<\aleph(\lambda)$. Therefore $f[D]$ is a dense subset of some subarc of $\lambda^\omega+1$. By Theorem 3, $\aleph(\lambda)\leqq\aleph(f[D])\leqq\aleph(D)$. This involves a contradiction.

Finally suppose $f$ is a nonconstant map from $\lambda^\omega+1$ to $I^\mu$. Let $M$ be as in Theorem 4. Select a point $x$ in $M\cap$ image $f$ which is not the image of 1. Then $f^{-1}(x)$ is a compact subset of $\lambda^\omega$. Thus $y=$ supremum $f^{-1}(x)$ is in $f^{-1}(x)$. By Theorem 2, there is a descending sequence $\{d_n:n<\omega\}$ in $\lambda^\omega$ with limit $y$. Let $X=\{f(d_n):n<\omega\}$. Then $x$ is a cluster point of $X$. By Theorem 4, $\aleph(X)\geqq\aleph(\mu)$. This involves a contradiction.

5. **Other rigid pairs.** If we assume the Generalized Continuum Hypothesis, there are two arcs of arbitrarily large cardinality that form a rigid pair.

THEOREM 7. *For each uncountable cardinal* $\aleph$, *there are two arcs each of cardinality* $2^{\aleph}$ *that form a rigid pair.*

PROOF. Let $\mu$ and $\lambda$ be the first ordinals such that $\aleph(\mu)=\aleph$ and $\aleph(\lambda)=2^{\aleph}$. $\aleph(\mu)=\sum_{\alpha<\mu} 2^{\aleph(\alpha)}$. Therefore $I^{\mu}$ and $\lambda^{\omega}+1$ are two rigid arcs. $\aleph(I^{\mu})=2^{\aleph_0\cdot\aleph}=2^{\aleph}$ and $\aleph(\lambda^{\omega}+1)=2^{\aleph\cdot\aleph_0}=2^{\aleph}$.

An arc is *strongly homogeneous* provided it is homeomorphic to each of its subarcs. For references and a discussion of strongly homogeneous arcs, see [2]. Every such arc has cardinality $2^{\aleph_0}$. Therefore no arc in one of the previous rigid pairs is strongly homogeneous. However if $\mu$ is any limit ordinal such that every final segment has order type $\mu$, then every subarc of $I^{\mu}$ contains a subarc homeomorphic to $I^{\mu}$.

*Problem* 1. Is there a rigid pair of strongly homogeneous arcs?

*Problem* 2. Are there three arcs such that any two form a rigid pair?

*Problem* 3. Given an arbitrary cardinal $\aleph$, are there $\aleph$ arcs such that any two form a rigid pair?

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