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A PRODUCT THEOREM FOR H-GROUP FIBRATIONS

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ABSTRACT. Let (E, p, B) and (E', p', B) be H-group fibrations over B with basic fibers F and F' respectively. If there are base point preserving fiber maps $f: E \rightleftarrows E' : g$ such that f is a fiber H-homomorphism, then $E \times F'$ and $E' \times F$ have the same homotopy type.

1. **Introduction.** Let (E, p, B) and (E', p', B) be fiber structures satisfying the weak covering homotopy property and having basic fibers F and F' respectively. If these fibrations have certain H-group properties and there are fiber maps $f: E \rightleftarrows E' : g$ such that f is an H-homomorphism, this paper shows that the product spaces $E \times F'$ and $E' \times F$ have the same homotopy type. If the restriction $f|_F : F \to F'$ is a homotopy equivalence, it follows that f is a fiber homotopy equivalence. This analogue of Dold's fiber homotopy equivalence theorem [**2**, Theorem 6.3] requires no local contractibility property of the base B.

Under more restrictive hypotheses it is proved that f is a fiber homotopy equivalence provided that it induces a homotopy equivalence on the spaces of based loops in E and E'.

2. H**-group fibrations.**

DEFINITION. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of topological spaces with base points and continuous maps is *exact* means that

(1) the composition gf is null-homotopic (i.e., homotopic to the constant map whose only value is the base point of Z); and

(2) for each space W and continuous map $h: W \to Y$ such that gh is null-homotopic, there is a continuous map $h_*: W \to X$ such that $fh_* \sim h$ (homotopic).

DEFINITION. A space X with continuous multiplication $\cdot$ is an *H-group* means that

(1) there is a point $x_0 \in X$ such that the maps defined by multiplication on the left and on the right by x_0 are homotopic to the identity map id_X on X;

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(2) the multiplication on X is homotopy associative; and

(3) there is a continuous map $j: X \to X$ such that the product maps $j \cdot \mathrm{id}_X$ and $\mathrm{id}_X \cdot j$ are homotopic to the constant map $c(x_0)$ whose only value is x_0.

The point x_0 is called a *homotopy unit* for X and j is called an *inversion*. If X and Y are H-groups, a continuous map $q: X \to Y$ is an *H-homomorphism* provided that the maps $(x_1, x_2) \to q(x_1 \cdot x_2)$ and $(x_1, x_2) \to q(x_1) \cdot q(x_2)$ from $X \times X$ to Y are homotopic.

NOTE. The functions involved in this paper are not assumed to be base point preserving unless specifically stated and it is not assumed that homotopies preserve base points at each level. All function spaces are assigned the compact open topology.

DEFINITION. Let $p: (E, e_0) \to (B, b_0)$ be a base point preserving map with the weak covering homotopy property and basic fiber $F = p^{-1}(b_0)$. Then (E, p, B) is an *H-group fibration* provided that E and F are H-groups with homotopy unit e_0, B is an H-group with homotopy unit b_0, and the inclusion $i: F \to E$ and the projection $p: E \to B$ are H-homomorphisms. If (E, p, B) and (E', p', B) are H-group fibrations over the same base B, a fiber map $f: E \to E'$ is a *fiber H-homomorphism* provided that f is an H-homomorphism from E into E' and the restriction $f' = f|_F : F \to F'$ is an H-homomorphism on the basic fibers.

For the remainder of this paper, (E, p, B) and (E', p', B) will denote H-group fibrations over the same space B with basic fibers $F = p^{-1}(b_0)$ and $F' = p'^{-1}(b_0)$ respectively where b_0 is the base point of B. In accordance with the definition, each of the H-groups E, F, E', F', and B has its base point as homotopy unit. For E and F the base point will be denoted by e_0, and for E' and F' the base point will be denoted by e_0'.

THEOREM 1. *Let (E, p, B) and (E', p', B) be H-group fibrations over B and $f: E \to E'$ a base point preserving fiber H-homomorphism. If there is a base point preserving fiber map $g: E' \to E$ then $E \times F'$ and $E' \times F$ have the same homotopy type.*

Theorem 1 will be established by applying the following result [**1**, Theorem 1] to a suitably chosen exact sequence:

THEOREM 2. *Suppose that the sequence*

$$0 \longrightarrow A \xrightarrow{q} C \xrightarrow{r} B$$

is exact, r has a right homotopy inverse $\tau: B \to C$, A and C are H-groups with base points as homotopy units, and $q: A \to C$ is an H-homomorphism. If the map $s: C \to B$ defined by

$$s(x) = r(x \cdot j\tau r(x)), \qquad x \in C,$$

where j is the inversion on C, is null-homotopic and the map $m: B\times A\to B$ defined by

$$m(b, a) = r(q(a)\cdot \tau(b)), \qquad (b, a)\in B\times A,$$

is homotopic to the projection Π_1 on the first component, then (C, r, B) and $(B\times A, \Pi_1, B)$ have the same homotopy type.

PROOF OF THEOREM 1. The desired sequence is

$$0 \longrightarrow F \xrightarrow{\ \theta\ } E\times F' \xrightarrow{\ \varphi\ } E'$$

where $\theta: F\to E\times F'$, $\varphi: E\times F'\to E'$ are defined by

$$\theta(x) = (i(x), f'(x)), \qquad \varphi(e, x') = ji'(x')\cdot f(e),$$
$$x\in F, \qquad (e, x')\in E\times F'.$$

Here $i': F'\to E'$ is the inclusion, $f': F\to F'$ is the restriction of f, $\cdot$ denotes the H-group operation, and j denotes inversion.

It is easily observed that the composition $\varphi\theta$ is null-homotopic. Some preliminary observations will be required before proving that the sequence has the required lifting properties.

Consider the sequence

$$\Omega E \xrightarrow{\ i\ } \Omega(E, F) \xrightarrow{\ m\ } F \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B$$

where ΩE is the space of based loops in E, $\Omega(E, F)$ is the space of paths in E with initial point e_0 and terminal point in F, m denotes evaluation at the terminal point and i denotes inclusion maps. This sequence and the corresponding sequence

$$\Omega E' \xrightarrow{\ i'\ } \Omega(E', F') \xrightarrow{\ m'\ } F' \xrightarrow{\ i'\ } E' \xrightarrow{\ p'\ } B$$

for (E', p', B) are exact.

Let PE denote the space of paths in E with initial point e_0, $\Pi: PE\to E$ the evaluation at the terminal point and

$$f_1: \Omega(E, F) \rightleftarrows \Omega(E', F'): g_1, \qquad \bar{f}: PE \rightleftarrows PE': \bar{g}$$

the maps defined from f and g by composition. Then $(PE, p\Pi, B)$ and $(PE', p'\Pi', B)$ have the weak covering homotopy property and the total spaces PE and PE' are contractible. It follows from [**2**, Theorem 6.1] that $\bar{f}$ is a fiber homotopy equivalence. Examination of the proof of that theorem shows that $\bar{g}$ is a fiber homotopy inverse for $\bar{f}$. In particular, the restrictions $f_1: \Omega(E, F) \rightleftarrows \Omega(E', F'): g_1$ are mutual homotopy inverses. This is not surprising since $\Omega(E, F)$ and $\Omega(E', F')$ are both homotopy equivalent to ΩB.

Suppose now that Y is a space and $h=(h_1, h_2): Y \to E\times F'$ is a continuous map such that φh is null-homotopic. Then $i'h_2$ is homotopic to fh_1 and

$$ph_1 = p'fh_1 \sim p'i'h_2 \sim 0.$$

Hence ph_1 is null-homotopic and there is a continuous map $t: Y\to F$ such that $it \sim h_1$. Then

$$i'(jh_2 \cdot f't) \sim ji'h_2 \cdot i'f't = ji'h_2 \cdot fit \sim ji'h_2 \cdot fh_1 \sim 0$$

and there is a continuous map $\gamma: Y \to \Omega(E', F')$ such that $m'\gamma$ is homotopic to $jh_2 \cdot f't$. The required homotopy lifting h_* of h is given by

$$h_* = t \cdot jmg_1\gamma : Y \to F.$$

To see this, observe that

$$\begin{aligned}\theta h_* &= (i[t\cdot jmg_1\gamma], f'[t\cdot jmg_1\gamma]) \sim (it\cdot jimg_1\gamma, f't\cdot jf'mg_1\gamma) \\ &\sim (it, f't\cdot jm'f_1g_1\gamma) \sim (h_1, f't\cdot jm'\gamma) \sim (h_1, h_2) = h.\end{aligned}$$

Suppose now that $k: X\to F$ is a map such that θk is null-homotopic. Then ik is null-homotopic so there is a continuous map $k': X\to\Omega(E, F)$ such that $mk' \sim k$. Then

$$m'f_1k' = f'mk' \sim f'k \sim 0$$

so there is a continuous map $k'': X \to \Omega E'$ such that $i'k'' \sim f_1k'$. Then g_2k'' maps X into ΩE where $g_2: \Omega E' \to \Omega E$ is the map induced by g. Hence

$$0 \sim mig_2k'' \sim mg_1i'k'' \sim mg_1f_1k' \sim mk' \sim k$$

and the indicated sequence is exact.

Consider the map $\sigma = fg\cdot j: E'\to E'$. Since f and g are fiber maps and p' is an H-homomorphism, then $p'\sigma$ is null-homotopic. Hence there is a homotopy lifting $\sigma_*: E' \to F'$ such that $i'\sigma_* \sim \sigma$. It follows that the map $\tau = (g, \sigma_*): E' \to E\times F'$ is a right homotopy inverse for φ. Although φ may not be an H-homomorphism with respect to the product H-group structure for $E\times F'$, the hypotheses of Theorem 2 are satisfied and Theorem 1 follows.

THEOREM 3. *Let (E, p, B) and (E', p', B) be H-group fibrations over B and $f: E\to E'$ a base point preserving fiber H-homomorphism. If the restriction $f': F\to F'$ is a homotopy equivalence and there is a base point preserving fiber map $g: E'\to E$, then f is a fiber homotopy equivalence.*

PROOF. Let $\tau = (g, \sigma_*)$ be the right homotopy inverse for φ given in the preceding proof and let $h': F'\to F$ be a homotopy inverse for f'.

Define $h = jih'\sigma_*\cdot g: E'\to E$. Then

$$fh \sim jfih'\sigma_* \cdot fg = ji'f'h'\sigma_* \cdot fg \sim ji'\sigma_* \cdot fg \sim j\sigma \cdot fg \sim \mathrm{id}_{E'}$$

so that h is a right homotopy inverse for f. To see that hf is homotopic to id_E, consider the map

$$s = (hf\cdot j, c(e_0')) : E \to E \times F'$$

where $c(e_0')$ is the constant map whose only value is e_0'. Then φs is null-homotopic so there is a continuous map $s_*: E \to F$ such that $\theta s_* \sim s$. Hence

$$\theta s_* = (is_*, f's_*) \sim s = (hf \cdot j, c(e_0')).$$

Then $f's_*$ is null-homotopic and, since f' is a homotopy equivalence, it follows that s_* is null-homotopic. Thus

$$hf\cdot j \sim is_* \sim 0$$

so hf is homotopic to the identity map on E. This completes the proof that f is a homotopy equivalence. Since f is a fiber map, the fact that it is a fiber homotopy equivalence follows from [**2**, Theorem 6.1].

Since the fiber map g of Theorems 1 and 3 is not required to bear any particular relationship to f, one might guess that these results would still be true without the existence of g. The following example shows that this is not the case.

EXAMPLE. Consider the Hopf map $h: S^3 \to S^2$ with basic fiber S^1. Since (S^3, h, S^2) is a regular Hurewicz fibration, then $(\Omega^2 S^3, \Omega^2 h, \Omega^2 S^2)$ is also a Hurewicz fibration. (Here $\Omega^2 = \Omega\Omega$ is the iteration of the based loop space functor.)

Let

$$(E, p, B) = (\Omega^2 S^3, \Omega^2 h, \Omega^2 S^2),$$
$$(E', p', B) = (\Omega^2 S^2 \times \Omega^2 S^1, \text{projection}, \Omega^2 S^2)$$

and let $f: E \to E'$ be the natural map induced by h. Observe that $F = F' = \Omega^2 S^1$, a contractible space. Since $\Omega^2 S^3$ and $\Omega^2 S^2$ are not homotopy equivalent, then $E \times F'$ and $E' \times F$ are not homotopy equivalent. Observe also that the restriction $f': F \to F'$ is a homotopy equivalence but $f: E \to E'$ is not a homotopy equivalence.

THEOREM 4. *Let (E, p, B) and (E', p', B) be H-group fibrations over B and $f: E \to E'$ a base point preserving fiber H-homomorphism. If the basic fibers F and F' are contractible in E and E' respectively and the induced map $f_2 = \Omega f: \Omega E \to \Omega E'$ is a homotopy equivalence, then the restriction $f': F \to F'$ is a homotopy equivalence.*

PROOF. Consider the diagram

$$(*)\qquad \begin{array}{ccccccc} \Omega E & \xrightarrow{i} & \Omega(E,F) & \xrightarrow{m} & F & \xrightarrow{i} & E \\ l_2\uparrow\downarrow f_2 & & l_1\uparrow\downarrow f_1 & & \downarrow f' & & \downarrow f \\ \Omega E' & \xrightarrow{i'} & \Omega(E',F') & \xrightarrow{m'} & F' & \xrightarrow{i'} & E' \end{array}$$

where

$$l_1:\Omega(E',F')\to\Omega(E,F),\qquad l_2:\Omega E'\to\Omega E$$

are homotopy inverses for f_1 and f_2 respectively. (Recall from the proof of Theorem 1 that f_1 is a homotopy equivalence.)

Since the horizontal sequences are exact and since F and F' are contractible in E and E' respectively, there exist continuous functions $\chi:F\to\Omega(E,F)$, $\chi':F'\to\Omega(E',F')$ such that

$$m\chi\sim \mathrm{id}_F,\qquad m'\chi'\sim \mathrm{id}_{F'}.$$

A straightforward argument involving the exactness and commutativity properties of diagram (*) shows that the functions

$$f_0=m'f_1\chi:F\to F',\qquad g_0=ml_1\chi':F'\to F$$

are mutual homotopy inverses. Since f' is homotopic to f_0, then f' is a homotopy equivalence.

REMARK. If F and F' are contractible in E and E' respectively, it follows from [**1**, Theorem 7] that we have homotopy equivalences

$$F\times\Omega E\approx\Omega B\approx F'\times\Omega E'.$$

With the algebraic properties of the spaces involved and the fact that the induced map $f_2:\Omega E\to\Omega E'$ is a homotopy equivalence, it is then natural to expect F and F' to be homotopy equivalent.

The following result is an immediate consequence of Theorems 3 and 4:

COROLLARY. *In addition to the hypotheses of the preceding theorem, suppose that there is a base point preserving fiber map $g:E\to E'$. Then f is a fiber homotopy equivalence.*

COROLLARY. *In addition to the hypotheses of the preceding theorem, suppose that B is arcwise connected and has a numerable covering $\{V_\lambda\}$ such that each inclusion $V_\lambda\to B$ is null-homotopic. Then f is a fiber homotopy equivalence.*

PROOF. By Theorem 4, $f': F \to F'$ is a homotopy equivalence. Since B is arcwise connected, the desired conclusion follows from [**2**, Theorem 6.3].

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