PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

A DECOMPOSITION THEOREM FOR CLOSED COMPACT CONNECTED P.L. n-MANIFOLDS

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ABSTRACT. Let M be a compact connected P.L. n-manifold without boundary. Then M is the union of 3 sets E_1, E_2 and F where E_i, $i=1, 2$, is homomorphic to the interior of an n-ball and F is the P.L. image of an $(n-1)$-sphere. Further each point of F is a limit point of E_1 and E_2.

It is well known that an n-sphere is the union of three disjoint sets E_1, E_2, F where E_1 and E_2 are topologically equivalent to an Euclidean n-space and F is topologically equivalent to an $(n-1)$-sphere. The set F can be thought of as an embedded $(n-1)$-sphere in an n-sphere. In this note it will be shown that every compact connected n-manifold has a very similar property. Namely,

THEOREM. *Every compact connected P.L. n-manifold, without boundary $n>0$, is the union of three disjoint sets E_1, E_2 and F where*

(i) *E_i is topologically equivalent to E^n,*

(ii) *F is the P.L. image of S^{n-1},*

(iii) *each point of F is a limit point of E_i, $i=1, 2$.*

PROOF. Let M be a closed compact connected combinatorial n-manifold of dimension $\geqq 3$. (The theorem is trivial for dimension 1 and an obvious modification of the proof for dimension $\geqq 3$ will yield a proof of the theorem for dimension 2.) Let T be some fixed triangulation that is the second barycentric subdivision of some triangulation of M. Choose some n-simplex σ in T. It was shown in [1] that there is a spine K of $M\backslash\text{Int}(\sigma)$ such that

(i) the closure of each component of each intrinsic skeleton is a combinatorial cell triangulated by the 3rd barycentric subdivision, T^3, of T,

(ii) if p is a vertex of K, then the closure of each component of $\text{St}(p, M)\backslash K$ is a cell,

(iii) K is of dimension $(n-1)$ at each point of K.

Received by the editors April 9, 1971.
AMS 1970 *subject classifications*. Primary 57C05.
Key words and phrases. P.L. manifold, spine, decomposition.

Let $C_{(n-1),j}$ and $C_{(n-2),j}$ be the components of the intrinsic $(n-1)$-skeleton and the intrinsic $(n-2)$-skeleton of K respectively. If $C_{(n-2),j}\subset \mathrm{Cl}(C_{(n-1),i})$, let $A_{i,j}$ be a polyhedral arc from a point $p_i\in\mathrm{Int}(\mathrm{Cl}(C_{(n-1),i}))$ to a point of $q_j\in\mathrm{Int}(\mathrm{Cl}(C_{(n-2),j}))$. Since dimension of $M\leqq 3$, it may be assumed that $\mathrm{Int}(A_{i,j})\cap\mathrm{Int}(A_{r,s})=\varnothing$ if $i\neq r$ or $j\neq s$ and that $\mathrm{Int}(A_{i,j})\subset \mathrm{Int}(\mathrm{Cl}(C_{(n-1),i}))$. Let H' be a maximal tree in the union of the $A_{i,j}$'s. Let $H\subset H'$ be a subtree such that if $q_j\in H$, q_j is of order at least 2. For each $q_j\in H$, let $\mathrm{St}(q_j M, T^5)=B_j$. Because of properties of K, $B_j\cap\mathrm{Cl}(C_{(n-1),i})$ is either an $(n-1)$-cell or empty. If $(B_j\cap C_{(n-1),i})\cap H$ contains an arc A such that $q_j\in A$ then call $\mathrm{Cl}(B_j\cap C_{(n-1),i})$ a good cell. Otherwise call $\mathrm{Cl}(B_j\cap C_{(n-1),i})$ a bad cell.

Let K' be K with the interiors of all bad cells deleted. Since $\mathrm{Cl}(B_j\cap C_{(n-1),i})$ is a cell, the closure of each component of the intrinsic $(n-1)$-skeleton of K' is a cell. Let K^* be the union of all the boundaries of all the closures of the components of the intrinsic $(n-1)$-skeleton of K'. Let $L=\mathrm{Cl}(K'\backslash N(K^*, M, T^5))$. ($N(K, L, T^i)$ denotes the union of the closed simplices of L triangulated by T^i that have a vertex in K.) Since L is the union of $(n-1)$-cells $L\cup H$ collapses to H. Let $(\bigcup B_i)\cup N(L, M, T^5)=Q$. By construction $M\backslash\mathrm{Int}(\sigma)$ collapses to Q. Since the neighborhood was taken in the 5th barycentric subdivision each point of K^* is accessible from σ. Hence the point set boundary of Q is the P.L. image of S^{n-1} and $M\backslash Q$ is homeomorphic to E^n. Since $L\cup H$ is collapsible, $\mathrm{Int}(Q)$ is homeomorphic to E^n and the theorem follows.

References

1. B. G. Casler and T. J. Smith, *Standard spines of compact connected combinatorial n-manifolds with boundary* (to appear).

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