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A NOTE ON ZERO DIVISORS IN GROUP-RINGS

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ABSTRACT. Let ZG_1 and ZG_2 be the integral group rings of groups G_1 and G_2 with a common normal subgroup H and let K be a subgroup of H. Let G be the free product of G_1 and G_2 amalgamating K. If ZG_1 and ZG_2 are integral domains and if ZH has the Ore condition then ZG is again an integral domain.

In this paper we apply a theorem of P. M. Cohn [**1**] to show that the group-ring of some generalized free products of groups has no zero divisors.

Recall that a ring R without zero divisors has the right Ore condition if any two nonzero elements of R have a common nonzero right multiple. In this situation R has a uniquely determined skew field D of right quotients: every element of D is of the form xy^{-1} with $x, y \in R$. (See e.g. [**2**, Theorem 1.3].)

Let us agree, by abus de language, to say that the group G has no zero divisors if the group-ring ZG has no zero divisors. Let H be a subgroup of the group G without zero divisors, and suppose ZG has the right Ore condition. (We note in passing that the right and left Ore condition are equivalent for group-rings.) Then ZH also has the right Ore condition. For ZG is a free right ZH module freely generated by a right transversal of G modulo H. Thus if x, y are in ZH there are nonzero elements t, u in ZG with $xt=yu$. We need only consider this equation coset per coset to find nonzero elements t' and u' in ZH with $xt'=yu'$. This said, we may proceed to our results.

THEOREM 1. *Let G_i $(i=1, 2)$ be a group without zero divisors, and let H_i be a normal subgroup of G_i such that ZH_i has the right Ore condition. Let K be a common subgroup of H_1 and H_2 and let G be the generalized free product of G_1 and G_2 amalgamating K. Then ZG has no zero divisors.*

PROOF. Let D_i be the fields of quotients of ZH_i, and consider the abelian group $R_i=ZG_i \otimes_{ZH_i} D_i$. We turn R_i into a ring by defining

$$(g_1 \otimes l_1^{-1})(g_2 \otimes l_2^{-1}) = g_1g_2 \otimes (g_2^{-1}l_1g_2)^{-1}l_2^{-1}$$

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for $g_1, g_2 \in G_i$, $l_1, l_2 \in ZH_i$, and extending by linearity. We leave it to the reader to verify that this is well defined. (Let $s \in ZH$, $r \in ZG$, $r = \sum \alpha_i g_i$. Then there are elements $x_i \in ZH$, such that $g_i^{-1} s g_i x_i = t$, for some $t \in ZH$. Thus $s \sum \alpha_i g_i x_i = \sum \alpha_i g_i g_i^{-1} s g_i x_i = \sum \alpha_i g_i t = rt$. Thus $ZH - \{0\}$ is a right divisor set in ZG (see again **[2]**) and by Ore's theorem ZG has a ring of right quotients with respect to ZH. Our definition gives a concrete representation for this ring of quotients.)

Since ZG_i is a free ZH_i module, the map $ZG_i \to ZG_i \otimes 1$ is a monomorphism. Further R_i has no zero divisors. For $(x_1 \otimes d_1^{-1})(x_2 \otimes d_2^{-1}) = 0$ only if $(1 \otimes d_1^{-1})(x_2 \otimes 1) = 0$ which forces $x_2 \otimes 1 = 0$ since $1 \otimes d_1^{-1}$ is invertible.

Now, as we pointed out, ZK has the Ore condition and its skew field D of quotients is contained in D_i. We now consider R_i as a right D vector space by identifying D with $1 \otimes D$. Let $\mathcal{S}_i \cup \{1\}$ be a right tranversal for K in G_i and suppose that $\sum_j (s_j \otimes 1) d_j = 0$ with the s_j distinct elements of $\mathcal{S}_1 \cup \{1\}$, say, and $d_i \in D$. Then, for some d_j' and d in ZK, $d_j = d_j' d^{-1}$, and thus $\sum_j (s_j \otimes 1) d_j' = \sum_j (s_j d_j' \otimes 1) = 0$. It follows that the d_j', and hence the d_j, are all zero. Thus $\mathcal{S}_i \cup \{1\}$ may be extended to a basis $B_i \cup \{1\}$ of R_i qua D vector space.

We now form the free product R (qua rings) of R_1 and R_2 amalgamating D. By Cohn's theorem, R has no zero divisors and the set B of monomials on the alphabet $B_1 \cup B_2$ (with consecutive letters in different factors) forms, together with 1, a D basis for R.

It remains to show that ZG is contained in R. To this effect we need only show that the normal forms $s_{i_1} s_{i_2} \cdots s_{i_k} k$ with $k \in K$ and no two $s_{i_j}, s_{i_{j+1}}$ in the same $\mathcal{S}_i$ and Z independent. This is however an immediate consequence of the D independence of B.

Group-rings with the Ore condition are fairly common, as the following shows:

PROPOSITION. *Let G be a solvable group without zero divisors. Then ZG has the (right) Ore condition.*

PROOF. It is clearly sufficient to prove the proposition for finitely generated G. Then there is a finite normal series with cyclic factors between $[G, G]$ and G. Using induction both on the length of this series and on the solvability length of G we may assume that G has a normal subgroup H with G/H cyclic and such that ZH has the Ore condition.

Let a generate G modulo H and let x and y be nonzero elements of ZG, say $x = \sum_{i=0}^{n} h_i a^i$, $y = \sum_{i=0}^{m} k_i a^i$ with $h_i, k_i \in ZH$ and $n \geqq m$. (Since a is a unit in ZG, it is clear that we need only consider elements involving positive powers of a.) By assumption there exist k_n' and h_m' with $h_n k_n' = k_m h_m'$. Let

M be the matrix

$$M = \begin{pmatrix} 1 & 0 \\ a^{-n}k'_n a^n & -a^{-m}h'_m a^m a^{n-m} \end{pmatrix}.$$

Then, if $\binom{x'}{y'} = M\binom{x}{y}$ we may assume by induction on $n+m$ that there is a row vector (t_1, t_2) with nonzero entries such that $(t_1, t_2)\binom{x'}{y'}=0$. If we set $(t'_1, t'_2)=(t_1, t_2)M$, then (t'_1, t'_2) is a nonzero vector with $(t'_1, t'_2)\binom{x}{y}=0$, as required.

CONJECTURE. If ZG has no zero divisors and does not have the Ore condition, then G contains a free (noncyclic) subgroup.

THEOREM 2. *Let G_i ($i=1, 2$) be a group such that ZG_i is embeddable in a skew field D_i and H be a subgroup of G_i such that ZH has the Ore condition. If G is the free product of G_1 and G_2 amalgamating H, then ZG has no zero divisors.*

PROOF. Both D_1 and D_2 contain the quotient skew field D of ZH and we may form the free product R of D_1 and D_2 amalgamating D. Proceeding as in the proof of Theorem 1, we find that ZG is contained in R which has no zero divisors.

The above results extend an unpublished theorem of G. Baumslag (who proved that the free product of two residually torsion free nilpotent groups amalgamating a cycle has no zero divisors) and overlap with some work of A. Karrass and D. Solitar (also unpublished) who proved that the free product of two locally indicable groups amalgamating a cycle is again locally indicable.

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