

QUOTIENT GROUPS OF TOPOLOGICAL GROUPS WITH NO SMALL SUBGROUPS

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ABSTRACT. It is shown here that a quotient group of a topological group with no small subgroups *can* have small subgroups.

It is well known that a quotient group of a Lie group is a Lie group, or equivalently ([2], [6]) that a quotient group of a locally compact group with no small subgroups is a locally compact group with no small subgroups. Irving Kaplansky [3] asks: if G is a topological group with no small subgroups and H is a closed normal subgroup of G , is G/H (necessarily) a group with no small subgroups? We answer the question in the negative here.

THEOREM. *If X is any metric space and $F(X)$ is the free abelian topological group on X [4], then $F(X)$ has no small subgroups.*

PROOF. By the Arens-Eells embedding theorem ([5], [1]) X can be embedded isometrically in a normed linear space N as a Hamel basis. Let S be the subgroup of (the additive group structure of) N generated by X . We claim that N , and hence S , has no small subgroups. This is seen by noting that the unit ball of N contains no nontrivial subgroups.

Since $F(X)$ is the free abelian topological group on X , there exists a continuous homomorphism f of $F(X)$ onto S which acts identically on X . Using the fact that X is a Hamel basis for N , we see that f is an algebraic isomorphism. Consequently $F(X)$ has no small subgroups.

EXAMPLE. Let X be the cartesian product of a countably-infinite family of topological groups, each of which is topologically isomorphic to the circle group with its usual topology. Clearly X is a topological group with small subgroups.

Since X is a metric space, by the above theorem, $F(X)$ has no small subgroups. Finally we note that, by Theorem 23 of [4], X is a quotient group of $F(X)$.

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