SPHERICAL IMAGES OF CONVEX SURFACES

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Let $x: M^n \rightarrow R^{n+1}$ be an isometric immersion of an orientable, complete riemannian manifold M^n (superscripts denote dimension) into the euclidean space R^{n+1} and let $\nu: M^n \rightarrow S^n \subset R^{n+1}$ be the normal (Gauss) map of x . Assume that the surface $x(M)$ is convex in R^{n+1} , i.e., $x(M) = \partial K$, where $K \subset R^{n+1}$ is a convex set with interior points. It has been long believed that the spherical image $\nu(M)$ is then convex in S^n (cf. [1, p. 25]). Recently, H. Wu [2] provided a counterexample to this assertion and proved the weaker statement:

THEOREM. *The closure of $\nu(M)$ is convex in S^n .*

Wu's proof is rather long and is designed to work in the more general context of continuous convex surfaces. We present here a simple proof for the differentiable case. We remark that the proof works for the infinite dimensional case as well.

PROOF OF THE THEOREM. Consider first the case where $\nu_0 = \nu(p_0)$ and $\nu_1 = \nu(p_1)$ are two nonantipodal points of S^n . Let ν be a point in the smallest arc of S^n joining ν_0 to ν_1 . Fix an orientation on M in such a way that the normals to $x(M)$ point towards K , and consider the *height function* $h: M \rightarrow R$ defined by $h(p) = \langle x(p), \nu \rangle$, $p \in M$. By convexity, h is bounded below. Let $p \in M$ and $\varphi(t)$ be the trajectory of $-\text{grad } h$ with $\varphi(0) = p$. Since $\|\text{grad } h\| \leq 1$ and M is complete, $\varphi(t)$ is defined for all $t \in (-\infty, \infty)$. We claim that $\|\text{grad } h(\varphi(t))\|$ is not bounded away from zero in $[0, \infty)$. Otherwise

$$\lim_{t \rightarrow \infty} h(\varphi(t)) - h(\varphi(0)) = - \int_0^\infty \|\text{grad } h(\varphi(t))\|^2 dt = -\infty,$$

which contradicts the fact that $h(\varphi(t))$ is bounded below.

It follows that either there is a critical point $\varphi(t_0)$ of h , $t_0 \in [0, \infty)$, and then the normal vector at $\varphi(t_0)$ is ν , or there exists a sequence of points in M whose normals converge to ν . In any case, ν belongs to the closure of $\nu(M)$.

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The remaining cases are simpler. If some or both v_0 and v_1 belong to the boundary of $v(M)$, and are not antipodal, we take the limit of minimal geodesics joining v_n^0 to v_n^1 , $n=1, \dots$, where $\{v_n^0\} \rightarrow v_0$, $\{v_n^1\} \rightarrow v_1$, and no pair v_n^0, v_n^1 is antipodal. If v_0 and v_1 belong to the closure of $v(M)$ and are antipodal, we consider a third point $v_2=v(p_2)$, distinct from v_0 and v_1 (which exists because M is connected), and prove easily that the smallest triangle $v_0v_1v_2$ belongs to the closure of $v(M)$.

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