ISOMETRIES OF H^p SPACES OF THE TORUS

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ABSTRACT. Denote by H^p ($1 \leq p \leq \infty$) the Banach spaces of complex-valued functions in L^p of the torus whose Fourier coefficients vanish off a half plane determined by a lexicographic ordering. The surjective isometries of the spaces H^p ($p \neq 2$) are characterized in terms of unimodular functions on the circle and conformal maps of the disc. For $1 < p < \infty$ ($p \neq 2$) the proof depends upon a characterization of certain invariant subspaces previously given by the authors.

Let A be the algebra of continuous complex-valued functions on $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ which are uniform limits of polynomials in λ . Denote by $H^p(d\theta)$ the closure of A in $L^p(d\theta)$ where $d\theta$ denotes normalized Lebesgue measure on the circle (norm closure for $1 \leq p < \infty$; w^* closure for $p = \infty$). It is well known that the Banach spaces $H^p(d\theta)$ may be identified with the Hardy classes by associating with each function in $H^p(d\theta)$ its analytic extension to the open unit disc via the Poisson integral.

DeLeeuw, Rudin, and Wermer [1], and independently Nagasawa [6], characterized the surjective isometries of $H^\infty(d\theta)$ and $H^1(d\theta)$. Forelli [2] extended the characterization to $H^p(d\theta)$ for $1 < p < \infty$, $p \neq 2$. We state their results in Propositions 1 and 2.

PROPOSITION 1. *A linear operator T of $H^\infty(d\theta)$ onto $H^\infty(d\theta)$ is an isometry if and only if*

$$(1) \quad (Tf)(\lambda) = \alpha f(\tau(\lambda)) \quad (f \in H^\infty(d\theta); |\lambda| = 1),$$

where α is a complex constant of modulus 1 and τ is a conformal map of the unit disc onto itself.

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PROPOSITION 2. Let $1 \leq p < \infty$, $p \neq 2$. A linear operator T of $H^p(d\theta)$ onto $H^p(d\theta)$ is an isometry if and only if

$$(2) \quad (Tf)(\lambda) = \alpha(\tau'(\lambda))^{1/p} f(\tau(\lambda)) \quad (f \in H^p(d\theta); |\lambda| = 1),$$

where α and τ are as in Proposition 1.

We denote by $A(T^2)$ the algebra of continuous, complex-valued functions on the torus $T^2 = \{(z, w) : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^m w^n$ where $(m, n) \in \mathcal{S} = \{(m, n) : n > 0\} \cup \{(m, 0) : m \geq 0\}$. Denoting normalized Haar measure on T^2 by dm , we define H^p as the closure of $A(T^2)$ in $L^p(dm)$ (norm closure for $1 \leq p < \infty$; w^* closure for $p = \infty$). The purpose of this paper is to present characterizations of the isometries of H^p onto H^p for $1 \leq p \leq \infty$, $p \neq 2$.

H^p consists of those functions in $L^p(dm)$ whose double Fourier coefficients vanish off the half-plane $\mathcal{S}$ which determines a lexicographic ordering. The maximal ideal space of $A(T^2)$ can be identified with $(\{z : |z| = 1\} \times \{w : |w| \leq 1\}) \cup (\{z : |z| \leq 1\} \times \{0\})$, with dm identified with $(z, w) = (0, 0)$. Since $A(T)^2$ is a logmodular algebra we have at our disposal the results of [4].

We denote by Z and W the functions $Z(z, w) = z$ and $W(z, w) = w$. The closure in $L^p(dm)$ of the polynomials in Z is denoted by Z^p ; the closure of the polynomials in $Z^m W^n$, $n \geq 1$, by I^p ; and finally the closure of the polynomials in Z and $\bar{Z}$ by L^p . By [5, Lemma 5, p. 467],

$$(3) \quad H^p = Z^p \oplus I^p$$

for $1 \leq p \leq \infty$, where $\oplus$ denotes algebraic direct sum. A function f in H^p is inner if $|f| \equiv 1$; f is outer if $f \cdot A(T^2)$ is dense in H^p .

THEOREM 1. A linear operator T of H^∞ onto H^∞ is an isometry if and only if

$$(4) \quad (Tf)(z, w) = \alpha f(\tau(z), w\sigma(z)) \quad (f \in H^\infty; |z| = |w| = 1),$$

where α is a complex constant of modulus 1, τ is a conformal map of the unit disc onto itself, and σ is a unimodular measurable function.

According to [1, Theorem 3, p. 695] it suffices to prove

THEOREM 2. A linear operator Ψ of H^∞ onto H^∞ is an algebra automorphism if and only if

$$(5) \quad (\Psi f)(z, w) = f(\tau(z), w\sigma(z)) \quad (f \in H^\infty; |z| = |w| = 1),$$

where τ and σ are as in Theorem 1.

LEMMA 1. *If Ψ is an algebra automorphism of H^∞ , then Ψ carries inner functions to inner functions, $\Psi Z^\infty = Z^\infty$, and $\Psi I^\infty = I^\infty$.*

PROOF. If F is inner but ΨF is not, then there exists $\varepsilon > 0$ such that $m(K) > 0$ where $K = \{x: |\Psi F(x)| < 1 - \varepsilon\}$. Choose $h \in H^\infty$ with $|h(x)| = 1$ on K and $|h(x)| = 1 - \varepsilon$ on $T^2 \setminus K$ [4, Theorem 5.9, p. 297]. If $\Psi g = h$, $\|Fg\|_\infty = 1$ but $\|\Psi(Fg)\|_\infty = \|(\Psi F)h\|_\infty \leq 1 - \varepsilon$. Thus ΨF is inner.

Let M be the closure of ΨI^∞ in $L^2(dm)$. M is clearly invariant under multiplication by functions in H^∞ and also by $\bar{V}$ where $V = \Psi Z$. For if $f \in I^\infty$, $fZ \in I^\infty$, so $\Psi f = \Psi(fZ)\Psi(Z)$ or $\Psi(fZ) = (\Psi f)(\bar{V})$.

If M has the form FH^2 for some inner function F , then $F \cdot \bar{V} \in M = FH^2$ so $\bar{V} \in H^2$. But $\Psi Z \in H^2$ so it is a constant. This contradicts the fact that Ψ is injective, so $M \subseteq I^2$ [4, p. 293]. It follows, using (3), that $\Psi I^\infty \subseteq I^\infty$. Applying the same argument to the automorphism Ψ^{-1} , we conclude that $\Psi I^\infty = I^\infty$.

To show that $\Psi Z^\infty = Z^\infty$, it suffices to show that $\Psi Z \in Z^\infty$. Write $f = \Psi Z$ and suppose $f = f_1 + f_2$ where $f_1 \in Z^\infty$ and $f_2 \in I^\infty$. Then $fI^\infty = \Psi(ZI^\infty) = I^\infty$, so $f_2 = fg$ for some $g \in I^\infty$. Thus $g = f_2 \bar{f} = (f - f_1) \bar{f} = 1 - f_1 \bar{f}$, which is orthogonal to I^∞ . Thus g and hence f_2 vanish.

LEMMA 2. *If $E_1 \in Z^\infty$ and $E_2 \in I^\infty$ are inner functions, and if for each Borel set $Y \subseteq T^2$, $\mu(Y) = m(X)$ where $X = \{(z, w): (E_1(z), E_2(z, w)) \in Y\}$, then $\mu \ll m$.*

PROOF. The Fourier-Stieltjes coefficients of μ are $\hat{\mu}(m, n) = \int \bar{E}_1^m E_2^n dm$. Thus $\hat{\mu}(m, 0) = a^m$ for $m \geq 0$ ($a = \int \bar{E}_1 dm$). Since $E_2 \in I^\infty$, $\hat{\mu}(m, n) = 0$ for $n \geq 1$. It follows that μ is the product measure $\mu = P dz \times dw = Q dm$, where $P(z) = (1 - |a|^2)/|1 - az|^2$, dz and dw are each Lebesgue measure, and $Q \in L^\infty(dm)$. In particular, if Y is m -null, then X is m -null. This argument is based on Forelli [2, p. 724].

PROOF OF THEOREM 2. By Lemma 1, $\Psi W \in I^\infty$. In fact $\Psi W = W\sigma$ for some $\sigma(z) = \sigma(z, w) \in L^\infty$, as can be shown by an argument similar to that by which we showed $\Psi Z \in Z^\infty$. Writing $\tau(z) = \tau(z, w) = (\Psi Z)(z, w)$, we see that τ is a conformal map of the disc by Proposition 1. Setting $E_1 = \tau$ and $E_2 = w\sigma$ in Lemma 2, we conclude that $f(\tau, w\sigma)$ is well defined for all measurable functions f . Thus (5) holds for all f in the algebra $\mathcal{A}$ generated by $Z^m W^n$, $(m, n) \in \mathcal{S}$.

To establish (5) for all $f \in H^\infty$, it suffices to show that the automorphism $\Phi f = \Psi^{-1}(f(\tau, w\sigma))$ is the identity. We have seen that $\Phi Z = Z$ and $\Phi W = W$ and the proof of Proposition 1 shows that Φ is the identity on Z^∞ . Thus it suffices to show that Φ is the identity on I^∞ .

First we show that $\Phi(\chi_K W) = \chi_K W$ where χ_K is a charactersitic function in L^∞ . Since the function $\Phi(\chi_K W)/W$ is equal to its own square, it too is a characteristic function $\chi_{K'} \in L^\infty$. There remains only to show that $K = K'$,

or in fact that $K \subseteq K'$ since the argument also applies to Φ^{-1} . If not, there exists a nonzero continuous function $f \in L^\infty$ with zero set $K_1 \subseteq K/K'$ of positive measure. Then

$$(6) \quad 0 = \Phi(fW)\Phi(\chi_{K_1}W) = fW\chi_{K_1}'W.$$

Since $K_1 \subseteq K$, $K_1' \subseteq K'$, so f does not vanish on K_1' . This contradicts (6).

Thus $\Phi(\chi_K W) = \chi_K W$, and in general for $g \in L^\infty$, $\Phi(gW^n) = gW^n$ ($n \geq 1$). If $g \in I^\infty$, $g = \sum_{i=1}^n g_i W^i + hW^n$ where $g_i \in L^\infty$ and $h \in I^\infty$. Since $\Phi g = \sum_{i=1}^n g_i W^i + (\Phi h)W^n$ where $\Phi h \in I^\infty$, the Fourier coefficients of g and Φg agree, so $\Phi g = g$.

REMARK. Using essentially the same argument we can show that the automorphisms of $A(T^2)$ are also given by (5) except that here σ is continuous. However, this can more easily be done by considering the homeomorphisms of the maximal ideal space of $A(T^2)$ induced by the automorphisms of the algebra.

THEOREM 3. A linear operator T of H^p onto H^p ($1 \leq p < \infty$, $p \neq 2$) is an isometry if and only if

$$(7) \quad (Tf)(z, w) = \alpha(\tau'(z))^{1/p} \cdot f(\tau(z), w\sigma(z)),$$

for all $f \in H^p$, where $|z| = |w| = 1$, α is a complex constant of modulus 1, τ is a conformal map of the unit disc onto itself, and σ is a unimodular measurable function on the circle.

The proof depends on our results in [5] on the characterization of sesqui-invariant subspaces of H^p . A closed subspace $M \subseteq H^p$ is called *invariant* if $fM \subseteq M$ for all $f \in H^\infty$. An invariant subspace $M \subseteq H^p$ is called *sesqui-invariant* if $ZM \subseteq M$ and *simply invariant* if this is not the case. If M is sesqui-invariant, it has the form

$$M = \chi_E \cdot \psi \cdot I^p$$

where ψ is unimodular and χ_E is the characteristic function of the support set of M ([5, Theorem 3, p. 471]; see also [5, p. 473 for the torus case]). If M is simply invariant, it has the form $M = \psi H^p$ (ψ unimodular) by the generalized Beurling theorem [8].

LEMMA 3. Let $F = T(1)$ and E be the support set of F . Then $m(E) = 1$.

PROOF. Since $F \in H^p$, χ_E is independent of w , so $G = w(1 - \chi_E) \in H^p$. Let $g = T^{-1}(G)$. Thus

$$\begin{aligned} \int |1 \pm g|^p dm &= \int |F \pm G|^p dm \\ &= \int |F|^p dm + \int |G|^p dm = 1 + \int |g|^p dm. \end{aligned}$$

Therefore

$$\int |1 + g|^p dm + \int |1 - g|^p dm = 2 \left[1 + \int |g|^p dm \right].$$

By [7, p. 275], $g=0$ a.e., so $m(E)=1$.

PROOF OF THEOREM 3. Lemma 3 insures that $dv=|F|^p dm$ and dm are mutually absolutely continuous. Using Forelli's argument [2, Proposition 2, p. 723] it follows that $Sf=Tf/F$ defines an isometry S of H^p into $L^p(dv)$ which takes the algebra $\mathcal{A}$ generated by $Z^m W^n$, $(m, n) \in \mathcal{S}$, multiplicatively into $L^\infty(dv)$. Write $E_1=S(Z)$ and $E_2=S(W)$. For $f \in \mathcal{A}$, we have

$$(8) \quad Tf(z, w) = F \cdot f(E_1, E_2).$$

We show that $E_1 \in Z^\infty$ and $E_2 \in I^\infty$. Since $F \in H^p$, the sesqui-invariant subspace generated by F has the form JI^p , where J is unimodular. Thus $F=JG$ where $G \in I^p$ and the sesqui-invariant subspace generated by G is I^p . For $f \in S(\mathcal{A})$, $WFf \in I^p$, and the property of G insures that $W^2 Jf \in I^p$. Thus the invariant subspace generated by $S(\mathcal{A})$ has the form ψI^p or ψH^p , ψ unimodular.

In the first case $f\psi W \in \psi I^p$, so $f \in L^p \oplus I^p$ for all $f \in S(\mathcal{A})$ and similarly for the second case. In particular E_1 and $E_2 \in L^\infty \oplus I^\infty$. The same argument applied to the algebra generated by $Z^m W^n$, $n \geq 1$, shows that $\bar{E}_1 \in L^\infty \oplus I^\infty$, so $E_1 \in L^\infty$.

We conclude that $F \notin I^p$ (otherwise T would map H^p onto I^p). Thus $\int \log |F| dm > -\infty$, so $F=JG$ where now J is inner and G is outer. Also $\{FE_1^m\}$, $m \geq 0$, generate a simply invariant subspace, so by the usual argument $E_1 \in Z^\infty$. Since G is outer, the invariant subspace N generated by $\{JE_1^m E_2^n\}$, $n > 0$, is contained in H^p . Since $N = \psi H^p$ would imply that $\bar{E}_1 \in H^p$, we have $N = \psi I^p$. Thus $N \subseteq I^p$ so $JE_2 \in I^p$. $J \in H^p$ but $J \notin I^p$ (because $F \notin I^p$), so $E_2 \in I^\infty$. Thus T takes I^p into I^p .

Thus using Lemma 2, $f(E_1, E_2)$ is well defined for all measurable functions f . The density of $\mathcal{A}$ in H^p , $1 \leq p < \infty$, insures that (8) holds for all $f \in H^p$. Imitating Forelli's argument [2, p. 726] one shows that the function Q constructed in the proof of Lemma 2 satisfies

$$(9) \quad \int_X |F|^p dm = \int_X 1/Q(E_1) dm$$

for all Borel sets $X \subseteq T^2$. Since T is surjective, both T and T^{-1} carry I^p into I^p , so that $TI^p = I^p$. Again using the argument of [2] (beginning at the bottom of p. 726) it follows that E_1 , considered as a function of z alone, is a.e. the boundary value function of a conformal map τ_1 of the disc onto itself. Define $\tau(z, w) = \tau_1(z)$. We have $|\tau'| = 1/Q(\tau)$ and (9) becomes

$$\int_X |F|^p dm = \int_X |\tau'| dm$$

for all Borel sets $X \subseteq T^2$. Thus F and $(\tau')^{1/p}$ have the same modulus. Since the latter is outer, we can show that they differ by a constant of modulus one by showing F is outer. If $F=JG$, J inner and G outer, then $GH^p = H^p = TH^p = FH^p$. Dividing by G , $H^p = JH^p$, so $J \equiv \alpha$, $|\alpha| = 1$.

To complete the proof it suffices to show that $E_2 = W\sigma$ where $\sigma \in L^\infty$. For this we need to show that $\sigma I^p = I^p$ (see the analogous argument for $p = \infty$). But since F and $1/F$ are bounded, $\sigma I^p = \overline{WS}(SW)(SI^p) = \overline{WS}(WI^p) = \overline{WS}I^p = I^p$.

For the case $p=1$, Theorem 3 can also be obtained by adapting the original argument of deLeeuw, Rudin and Wermer [1, Theorem 2, p. 694] in which they deduce the isometries of $H^1(d\theta)$ by exploiting the special properties of the extreme points of the unit ball of $H^1(d\theta)$. To do this one needs three facts about functions on the torus: (a) the extreme points of the unit ball of H^1 are the outer functions of norm one (Gamelin [3]), (b) the identity $\int f dm = \int f(\tau, w\sigma)\tau' dm$ (a straightforward calculation), and (c) the result of Lemma 4 below. Let B^e be the set of extreme points in the unit ball of H^1 , $P(m) = \{z: |z| < 1\} \times \{0\}$, and $D_z = \{z\} \times \{w: |w| < 1\}$ for each $|z|=1$.

LEMMA 4. *A function $f \in H^1$ of norm 1 lies in the closure of B^e if and only if*
 (10) *$\hat{f}$ has no zeros on $P(m)$ and $\hat{f}$ has no zeros on D_z for almost all z .*

PROOF. If f lies in the closure of B^e , then there exist $f_n \in B^e$ converging uniformly on compact sets to $\hat{f}$ on $P(m)$ and on each D_z for almost all z . (10) follows.

Conversely suppose (10) holds. Define $f_r(z, w) = f(z, rw)$, $0 < r < 1$. Let $f_r = F_r g_r$, F_r inner, g_r outer. One shows that F_r is independent of r , say $F_r = F \in Z^\infty$. Let $h_r(z, w) = F(rz)$. Then f is the L^1 limit of the outer functions $h_r g_r$, so f lies in the closure of B^e .

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