APPROXIMATION OF COMMUTING TRANSFORMATIONS

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ABSTRACT. Let σ and τ be two (measure-preserving) transformations. The main purpose of the paper is to show that if τ admits approximation by partitions and that if σ commutes with a power τ^s of τ , then σ can be approximated by a finite number of powers of τ . As an application of the result we solve a problem posed earlier, showing that there exist strongly mixing transformations with only a finite number of prescribed roots.

1. Introduction. Let σ and τ be two (measure-preserving) transformations of a normalized nonatomic Lebesgue space. The main purpose of this paper is to show that if τ admits an approximation by partitions (Definition 2.5) and if σ commutes with a power τ^s of τ , then σ can be approximated by a finite combination of powers of τ (Theorem 3.1). This result generalizes the main theorems of [1] and [2], which correspond to the special case of $s=1$. The methods are extensions of those introduced in [2] and [3]. As an application, we show that there are strongly mixing transformations with only a finite number of prescribed roots, answering a question raised in [4, p. 133].

2. Definitions. Let $(X, \mathcal{F}, \mu)$ be a normalized nonatomic Lebesgue space; i.e. a measure space isomorphic to the unit interval. All subsets of X that are referred to are understood to be in $\mathcal{F}$. In what follows we will recall a few definitions from [1] and [2] to make the present paper self-contained.

DEFINITION 2.1. A collection ξ of pairwise disjoint sets will be called a partition. The union of the sets in ξ is denoted by X_ξ , which, in general, is a proper subset of X . If ξ is a partition, then $\mathcal{F}(\xi)$ is the σ -field of subsets of X_ξ generated by the sets in ξ . If $\xi = \{A_0, A_1, \dots\}$ is a (finite or) countable partition, then any set $A \in \mathcal{F}(\xi)$ can be written as $A = \sum_{i=0}^{\infty} a_i A_i$ where a_i is zero or one for each $i=0, 1, \dots$, with the convention that $0 \cdot B = \emptyset$ and $1 \cdot B = B$ for any set B .

DEFINITION 2.2. If ξ is a countable partition and if $A \in \mathcal{F}$, then among the sets in $\mathcal{F}(\xi)$ there is at least one whose symmetric difference with A has minimal measure. We denote by $A(\xi)$ any one of these sets.

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DEFINITION 2.3. Let $F_n, F \in \mathcal{F}$. We write

$$\lim_{n \rightarrow \infty} F_n = F \quad \text{if} \quad \lim_{n \rightarrow \infty} \mu(F_n \triangle F) = 0.$$

Let ξ_n be a sequence of countable partitions. We write

$$\xi_n \rightarrow \varepsilon \quad \text{if} \quad \lim_{n \rightarrow \infty} A(\xi_n) = A$$

for each $A \in \mathcal{F}$.

DEFINITION 2.4. In what follows, by a transformation τ (of X) we mean an invertible mapping $\tau: X \rightarrow X$ so that both τ and τ^{-1} are measure preserving. Let τ be a transformation and ξ be a partition. Then ξ is said to be τ -admissible if $\xi = \{A_0, A_1, \dots, A_q\}$ is a finite partition and if $\tau A_{i-1} = A_i$ for all $i = 1, \dots, q$.

DEFINITION 2.5. A transformation τ is said to admit an approximation by partitions, if there exists a sequence $\{\xi_n\}$ of τ -admissible partitions so that $\xi_n \rightarrow \varepsilon$.

DEFINITION 2.6. If $A, B \in \mathcal{F}$ and $\varepsilon > 0$, then $A = B + E(\varepsilon)$ means that $\mu(A \triangle B) < \varepsilon$.

3. Approximation of commuting transformations. Let τ and σ be two transformations and let $s \geq 1$ be an integer so that τ^s and σ commute; i.e., $\tau^s \sigma x = \sigma \tau^s x$ for almost all $x \in X$. Also, assume that τ admits an approximation by partitions.

LEMMA 3.1. Let $\xi = \{A_0, A_1, \dots, A_q\}$ be a τ -admissible partition. Assume that there is an integer $p \geq 0$ so that $q = (p+1)s$. Let $A, B \in \mathcal{F}(\xi)$, $B \cap A_0 = \emptyset$ and $\sigma A = B + E(\varepsilon)$. Then there are $2s$ sets $F^r, G^r, r = 1, \dots, s$, and $2s$ integers $m^r, n^r, r = 1, \dots, s$, so that:

- (1) for each $r = 1, \dots, s$, $m^r, n^r \geq 0$, $m^r + n^r \geq q + 1$,
- (2) for each $r = 1, \dots, s$, $F^r \cap G^r = \emptyset$ and

$$\mu(F^r) \leq \frac{n^r + 2s}{q + 1} \frac{1}{s},$$

$$\mu(G^r) \leq \frac{m^r + 2s}{q + 1} \frac{1}{s},$$

- (3) $\sigma A = \sum_{r=1}^s \{\tau^{m^r}(A \cap F^r) + \tau^{-n^r}(A \cap G^r)\} + E(2\varepsilon)$.

PROOF. Let $C_i^r = A_r \cap \sigma \tau^i A_0$. If $x \in C_i^r \cap C_j^r$ then there exist two points $y, z \in A_0$ so that $x = \sigma \tau^i y = \sigma \tau^j z$ which implies that $\tau^{i-j} y = z$. Hence either $i = j$ or $|i - j| \geq q + 1$. Therefore, for a fixed r , any $q + 1$ consecutive C_i^r 's are pairwise disjoint.

Now, let $R = X - \bigcup_{i=1}^q A_i$ and consider a fixed l , $0 \leq l \leq q$. Then

$$\begin{aligned}\sigma A_l &= \sum_{r=1}^s \sum_{k=0}^p A_{ks+r} \cap \sigma A_l + R \cap \sigma A_l \\ &= \sum_{r=1}^s \sum_{k=0}^p [(\tau^{ks} A_r) \cap (\sigma \tau^l A_0)] + R \cap \sigma A_l.\end{aligned}$$

Since τ^{ks} commutes with σ , we then have that

$$\sigma A_l = \sum_{r=1}^s \sum_{k=0}^p \tau^{ks} [A_r \cap \sigma \tau^{l-ks} A_0] + R \cap \sigma A_l$$

or that

$$\sigma A_l = \sum_{r=1}^s \sum_{k=0}^p \tau^{ks} C_{l-ks}^r + R \cap \sigma A_l.$$

Hence, if $A = \sum_{l=0}^q a_l A_l$, $B = \sum_{l=0}^q b_l A_l$, then

$$\begin{aligned}\sigma A &= \sum_{l=0}^q a_l \sigma A_l \\ &= \sum_{l=0}^q \sum_{k=0}^p \sum_{r=1}^s a_l \tau^{ks} C_{l-ks}^r + R \cap \sigma A \\ &= \sum_{k=0}^p \sum_{r=1}^s b_{ks+r} A_{ks+r} + E(\varepsilon) \\ &= \sum_{k=0}^p \sum_{r=1}^s b_{ks+r} \tau^{ks} A_r + E(\varepsilon)\end{aligned}$$

or, since $B \cap R = \emptyset$,

$$\sum_{k=0}^p \tau^{ks} \sum_{r=1}^s \sum_{l=0}^q a_l C_{l-ks}^r = \sum_{k=0}^p \tau^{ks} \sum_{r=1}^s b_{ks+r} A_r + E(\varepsilon).$$

Now, for a fixed $k=0, \dots, p$ and $r=1, \dots, s$, the sets C_{l-ks}^r , $l=0, \dots, q$, are pairwise disjoint. Hence, the characteristic function of $\sum_{l=0}^q a_l C_{l-ks}^r$ is $\sum_{l=0}^q a_l \psi_{l-ks}^r$, where ψ_{l-ks}^r denotes the characteristic function of C_{l-ks}^r . Hence

$$\sum_{r=1}^s \int_{A_r} \sum_{k=0}^p \left| \sum_{l=0}^q a_l \psi_{l-ks}^r - b_{ks+r} \right| d\mu < \varepsilon.$$

Therefore there exist $x_r \in A_r$, $r=1, \dots, s$, so that

$$\sum_{r=1}^s \sum_{k=0}^p \left| \sum_{l=0}^q a_l \psi_{l-ks}^r(x_r) - b_{ks+r} \right| \mu(A_0) < \varepsilon.$$

Hence, if

$$D = \sum_{r=1}^s \sum_{k=0}^p \sum_{l=0}^q a_l \psi_{l-ks}^r(x_r) A_{ks+r}$$

then $\sigma A = D + E(2\varepsilon)$.

Now to define the integers m^r , n^r and the sets F^r , G^r consider a fixed r , $1 \leq r \leq s$. Then, by a change of the summation indices one can see that

$$\begin{aligned} \sum_{k=0}^p \sum_{l=0}^q a_l \psi_{l-ks}^r(x_r) A_{ks+r} &= \sum_{u=-ps}^{(p+1)s} \psi_u^r(x_r) \sum_{k=0; 0 \leq u+ks \leq q}^p a_{u+ks} A_{r+ks} \\ &= \sum_{u=-ps}^{r-1} \psi_u^r(x_r) \sum_{k=0; 0 \leq u+ks}^p a_{u+ks} A_{r+ks} \\ &\quad + \sum_{u=r}^{(p+1)s} \psi_u^r(x_r) \sum_{k=0; u+ks \leq q}^p a_{u+ks} A_{r+ks} \\ &= \sum_{u=-ps}^{r-1} \psi_u^r(x_r) \tau^{+(r-u)} \sum_{k=0; 0 \leq u+ks}^p a_{u+ks} A_{u+ks} \\ &\quad + \sum_{u=r}^{(p+1)s} \psi_u^r(x_r) \tau^{-(u-r)} \sum_{k=0; u+ks \leq q}^p a_{u+ks} A_{u+ks}. \end{aligned}$$

Each of these two sums with respect to u contains at most one nonzero term. If one of them contains all zero terms then we can define $F^r = \emptyset$ or $G^r = \emptyset$ or $F^r = G^r = \emptyset$ accordingly and see that all the conclusions of the lemma are satisfied. If both of them contain nonzero terms, corresponding, say to $u=v^r$ and $u=w^r$ with $-ps \leq v^r < r$, $r \leq w^r \leq (p+1)s$, we then let

$$F^r = \sum_{k=0; 0 \leq v^r+ks}^p A_{v^r+ks}, \quad G^r = \sum_{k=0; w^r+ks \leq q}^p A_{w^r+ks},$$

and $m^r = r - v^r$, $n^r = w^r - r$. Since $w^r - v^r \geq q+1$ it is clear that $m^r + n^r \geq q+1$, and $m^r, n^r \geq 0$, and that $F^r \cap G^r = \emptyset$. Also, we have that

$$\sigma A = \sum_{r=1}^s [\tau^{m^r}(A \cap F^r) + \tau^{-n^r}(A \cap G^r)] + E(2\varepsilon).$$

Now, to estimate the measure of F^r and G^r , first note that the measure of each A_l is not greater than $1/(q+1)$. Hence, for example, $\mu(G^r) \leq \kappa/(q+1)$, where κ is the number of integers in $\{k | 0 \leq k \leq p, w^r+ks \leq q\}$. Then

$$\kappa \leq \frac{q - w^r}{s} + 1 = \frac{q - n^r - r}{s} + 1 \leq \frac{1 - r + m^r}{s} + 1.$$

Hence $\kappa \leq m^r/s + 2$. Therefore $\mu(G^r) \leq ((m^r + 2s)/(q+1))(1/s)$. Similarly $\mu(F^r) \leq ((n^r + 2s)/(q+1))(1/s)$.

THEOREM 3.1. *There exist $2s^2$ sequences of sets $\{Y_v^{rt}\}$, $\{Z_v^{rt}\}$ and $2s^2$ sequences of integers $\{m_v^{rt}\}$, $\{n_v^{rt}\}$, where $v=1, 2, \dots$ and $r=1, \dots, s$; $t=1, \dots, s$, so that*

$$\sigma A = \lim_{v \rightarrow \infty} \sum_{t=1}^s \sum_{r=1}^s [\tau^{m_v^{rt}}(A \cap Y_v^{rt}) + \tau^{-n_v^{rt}}(A \cap Z_v^{rt})] \quad \text{for every } A \in \mathcal{F}.$$

Furthermore, for each $r, t=1, \dots, s$ one of the following 3 conditions holds:

- (1) $\lim_{v \rightarrow \infty} \mu(Z_v^{rt})=0$ and $m_v^{rt}=m^{rt}$ does not depend on v ,
- (2) $\lim_{v \rightarrow \infty} \mu(Y_v^{rt})=0$ and $n_v^{rt}=n^{rt}$ does not depend on v ,
- (3) $\lim_{v \rightarrow \infty} m_v^{rt}=\lim_{v \rightarrow \infty} n_v^{rt}=\infty$.

PROOF. Let $\xi_v=\{A_0(v), \dots, A_{q(v)}(v)\}$, $\xi_v \rightarrow \varepsilon$, be a sequence of τ -admissible partitions. Note that $q(v) \rightarrow \infty$ as $v \rightarrow \infty$. Without loss of generality, we will assume that for each $v=1, 2, \dots$ there exists an integer $p(v)$ so that $q(v)=[p(v)+1]s$. Choose a sequence $\varepsilon_v > 0$ so that $\varepsilon_v q(v) \rightarrow 0$ as $v \rightarrow \infty$. For each integer v we can find another integer $\lambda=\lambda(v)$ with the following properties:

(a) there exist $B_v^t, D_v^t \in \mathcal{F}(\xi_{\lambda(v)})$ so that $A_t(v)=B_v^t+E(\varepsilon_v)$, $D_v^t \cap A_0(\lambda(v))=\emptyset$ and $\sigma A_t(v)=D_v^t+E(\varepsilon_v)$ for every $t=1, 2, \dots, s$.

(b) $q(v) \leq \varepsilon_v q(\lambda(v))$.

We then have that $\sigma B_v^t=D_v^t+E(2\varepsilon_v)$. Now choose $F_v^{rt}, G_v^{rt}, m_v^{rt}$ and n_v^{rt} according to the previous lemma. Hence

$$\sigma B_v^t = \sum_{r=1}^s [\tau^{m_v^{rt}}(B_v^t \cap F_v^{rt}) + \tau^{-n_v^{rt}}(B_v^t \cap G_v^{rt})] + E(4\varepsilon_v),$$

or

$$\sigma A_t(v) = \sum_{r=1}^s [\tau^{m_v^{rt}}(A_t(v) \cap F_v^{rt}) + \tau^{-n_v^{rt}}(A_t(v) \cap G_v^{rt})] + E((6+s)\varepsilon_v).$$

We now let

$$Y_v^{rt} = \sum_{k=0}^{p(v)} \tau^{ks}(A_t(v) \cap F_v^{rt}), \quad Z_v^{rt} = \sum_{k=0}^{p(v)} \tau^{ks}(A_t(v) \cap G_v^{rt}).$$

Hence, if $A' = \sum_{l=0}^{q(v)} a_l A_l(v) \in \mathcal{F}(\xi_v)$ then

$$A = \sum_{k=0}^{p(v)} \sum_{t=1}^s a_{t+ks} A_{t+ks}(v)$$

agrees with A' on all $A_l(v)$, $l=1, \dots, q(v)$, except on $A_0(v)$. Hence $A=A'+E(1/q(v))$. But

$$\begin{aligned} \sigma A &= \sum_{t=1}^s \sum_{k=0}^{p(v)} a_{t+ks} \tau^{ks} \sigma A_t(v) \\ &= \sum_{t=1}^s \sum_{k=0}^{p(v)} a_{t+ks} \tau^{ks} \sum_{r=1}^s [\tau^{m_v^{rt}}(A_t(v) \cap F_v^{rt}) + \tau^{-n_v^{rt}}(A_t(v) \cap G_v^{rt})] \\ &\quad + E((6+s)q(v)\varepsilon_v) \\ &= \sum_{t=1}^s \sum_{r=1}^s [\tau^{m_v^{rt}}(A \cap Y_v^{rt}) + \tau^{-n_v^{rt}}(A \cap Z_v^{rt})] + E((6+s)q(v)\varepsilon_v) \end{aligned}$$

since, for example,

$$\begin{aligned}\sum_{k=0}^{p(v)} a_{t+ks} \tau^{ks} \tau^{m_v r_t} (A_t(v) \cap F_v^{r_t}) &= \tau^{m_v r_t} \sum_{k=0}^{p(v)} a_{t+ks} \tau^{ks} (A_t(v) \cap F_v^{r_t}) \\ &= \tau^{m_v r_t} (A \cap Y_v^{r_t}).\end{aligned}$$

Hence,

$$\begin{aligned}\sigma A' &= \sum_{t=1}^s \sum_{r=1}^s \{ \tau^{m_v r_t} (A' \cap Y_v^{r_t}) + \tau^{-n_v r_t} (A' \cap Z_v^{r_t}) \} \\ &\quad + E[(6+s)q(v)\varepsilon_v + (s^2+1)/q(v)].\end{aligned}$$

It is clear that this proves the first part of the theorem, the approximation of σA .

To complete the proof of the theorem assume that condition (3) in the statement of this theorem is not satisfied. Then either $m_v^{r_t}$ or $n_v^{r_t}$ repeats an integer infinitely many times. By considering a subsequence we may assume that, for example, $m_v^{r_t} = m^{r_t}$ for all $v=1, 2, 3, \dots$. But, since $F_v^{r_t}$ and $G_v^{r_t}$ are chosen with respect to the partition $\xi_{\lambda(v)}$, Lemma 3.1 implies that

$$\mu(G_v^{r_t}) \leq \frac{m^{r_t} + 2s}{q(\lambda(v)) + 1} \frac{1}{s}$$

and hence that

$$\begin{aligned}\mu(Z_v^{r_t}) &= \mu\left(\sum_{k=0}^{p(v)} \tau^{ks} (A_t(v) \cap G_v^{r_t})\right) \\ &\leq (p(v) + 1) \frac{m^{r_t} + 2s}{q(\lambda(v)) + 1} \frac{1}{s} \\ &\leq \frac{m^{r_t} + 2s}{s^2} \frac{q(v)}{q(\lambda(v))}\end{aligned}$$

or, finally that $\mu(Z_v^{r_t}) \leq ((m^{r_t} + 2s)/s^2)\varepsilon_v$. Hence $\lim_{v \rightarrow \infty} \mu(Z_v^{r_t}) = 0$.

This completes the proof of the theorem.

4. Mixing transformations with given roots. We will now apply Theorem 3.1 to show that there are strongly mixing transformations having only a finite number of specified roots. First we will note a few elementary results on transformations.

LEMMA 4.1. *Let τ and σ be two transformations of X . Then there exists a set $A \in \mathcal{F}$ so that $\tau(A \cap E) = \sigma(A \cap E)$ for all $E \in \mathcal{F}$ and so that $B = X - A$ is a (finite or) countable union of pairwise disjoint sets B_i , satisfying $(\tau B_i) \cap (\sigma B_i) = \emptyset$ for each $i=1, 2, \dots$. Furthermore, if λ is another transformation that commutes with both τ and σ then A is invariant under λ .*

PROOF. For each $E \in \mathcal{F}$ let A_E be the class of all measurable subsets F of E , for which $(\tau F) \cap (\sigma F) = \emptyset$, and let $\alpha_E = \sup\{\mu(F) | F \in A_E\}$. If $E \in \mathcal{F}$ has a subset G for which $\mu((\tau G) \triangle (\sigma G)) > 0$ then $F = \tau^{-1}(\tau G - \sigma G) \in A_E$ and $\mu(F) > 0$. Therefore either $\alpha_E > 0$ or $\tau G = \sigma G$ for all subsets G of E .

Now construct a sequence $\{B_1, B_2, \dots\}$ of sets as follows, using the axiom of choice.

Let B_1 be any set in A_X for which $\mu(B_1) \geq \frac{1}{2}\alpha_X$. If $B_1, B_2, \dots, B_n$ are already chosen let B_{n+1} be any set in $A_{X - \bigcup_{i=1}^n B_i}$ with $\mu(B_{n+1}) \geq \frac{1}{2}\alpha_{X - \bigcup_{i=1}^n B_i}$. Then B_n 's are pairwise disjoint and $(\tau B_n) \cap (\sigma B_n) = \emptyset$ for each $n = 1, 2, \dots$. Let $B = \bigcup_{n=1}^{\infty} B_n$ and $A = X - B$. We will now show that $\alpha_A = 0$. In fact, since $A \subset X - \bigcup_{i=1}^n B_i$ for all $n \geq 1$, $\alpha_A \leq \alpha_{X - \bigcup_{i=1}^n B_i} \leq 2\mu(B_{n+1})$ and $\mu(B_n) \rightarrow 0$, for B_n 's are pairwise disjoint and $\mu(X) < \infty$.

Now let λ be a transformation which commutes with τ and σ . Let $A' = \lambda A$. Then $\tau(A' \cap E) = \tau\lambda(A \cap \lambda^{-1}E) = \lambda\tau(A \cap \lambda^{-1}E) = \lambda\sigma(A \cap \lambda^{-1}E) = \sigma(A' \cap E)$ for any $E \in \mathcal{F}$. Hence $\alpha_{A'} = 0$. But this implies that $\mu(B \cap A') = 0$, since any nonnull subset of B has a positive α . Therefore A is invariant under λ .

LEMMA 4.2. *Let τ and σ be two transformations that satisfy the conditions in §3. Assume that τ^s is ergodic. Then there exists an approximation of σ as given in Theorem 3.1, for which either $\lim_{v \rightarrow \infty} m_v^r = \lim_{v \rightarrow \infty} n_v^r = \infty$ for each $r, t = 1, \dots, s$ or $\sigma = \tau^p$ for some integer p .*

PROOF. Consider the approximation of σ as given in Theorem 3.1 and assume that $\lim_{v \rightarrow \infty} m_v^r = \lim_{v \rightarrow \infty} n_v^r = \infty$ is not satisfied for some r_0, t_0 . Then either (1) or (2) of Theorem 3.1 holds. We will only consider the case where (1) holds; the other case is similar. Hence $m_v^{r_0 t_0} = p$ for some integer p and $\lim_{v \rightarrow \infty} \mu(Z_v^{r_0 t_0}) = 0$. If $\lim_{v \rightarrow \infty} \mu(Y_v^{r_0 t_0}) = 0$, then the term corresponding to $r_0 t_0$ does not play any role in the approximation of σ and the corresponding powers $m_v^{r_0 t_0}, n_v^{r_0 t_0}$ can be changed arbitrarily without disturbing the form of approximation. Hence, in this case we can find a new approximation for σ for which $\lim_{v \rightarrow \infty} m_v^{r_0 t_0} = \lim_{v \rightarrow \infty} n_v^{r_0 t_0} = \infty$. Now assume that $\limsup_{v \rightarrow \infty} \mu(Y_v^{r_0 t_0}) > 0$. In this case we will show that $\sigma = \tau^p$.

In fact, let A and $B = \bigcup_{n=1}^{\infty} B_n$ be the decomposition of X as given in Lemma 3.1, corresponding to the automorphisms τ^p and σ . Since τ^s commutes both with τ^p and σ , Lemma 4.1 gives that A is invariant under τ^s . Hence either $A = X$ or $B = X$, for τ^s is ergodic. If $A = X$, then $\sigma E = \tau^p E$ for all $E \in \mathcal{F}$ and hence $\sigma = \tau^p$ as is well known.

If $B = X$, since $\limsup_{v \rightarrow \infty} \mu(Y_v^{r_0 t_0}) > 0$, then there exists a $C = B_i$, so that $\limsup_{v \rightarrow \infty} \mu(Y_v^{r_0 t_0} \cap C) > 0$ and $(\sigma C) \cap (\tau^p C) = \emptyset$. But then it is clear that

$$\sum_{r=1}^s \sum_{t=1}^s [\tau^{m_v r t}(C \cap Y_v^{r t}) + \tau^{-n_v r t}(C \cap Z_v^{r t})]$$

fails to converge to σC . This is a contradiction and completes the proof.

To proceed, we will now quote a result of Ornstein [5], without proof.

THEOREM 4.1. *There exists a strongly mixing automorphism that admits an approximation by partitions.*

Recall that an automorphism τ of X is called strongly mixing if $\lim_{v \rightarrow \infty} (\tau^v E \cap F) = \mu(E)\mu(F)$ for each $E, F \in \mathcal{F}$.

THEOREM 4.2. *Let τ be a strongly mixing automorphism that admits an approximation by partition and let $s \geq 1$ be an integer. If an automorphism σ commutes with τ^s then σ is a power of τ ; i.e., there exists an integer p so that $\sigma = \tau^p$.*

PROOF. We apply Lemma 4.2. If σ is not a power of τ , then

$$\sigma A = \lim_{v \rightarrow \infty} \sum_{t=1}^s \sum_{r=1}^s [\tau^{m_v^{rt}}(A \cap Y_v^{rt}) + \tau^{-n_v^{rt}}(A \cap Z_v^{rt})] \quad \text{for all } A \in \mathcal{F},$$

where $\lim_{v \rightarrow \infty} m_v^{rt} = \lim_{v \rightarrow \infty} n_v^{rt} = \infty$ for each $r, t = 1, \dots, s$. Hence

$$\sigma A \subset \sum_{t=1}^s \sum_{r=1}^s [\tau^{m_v^{rt}} A + \tau^{-n_v^{rt}} A] + E_v \quad \text{where } \mu(E_v) \rightarrow 0 \text{ as } v \rightarrow \infty.$$

Therefore

$$\begin{aligned} \sigma A \subset \sum_{r,t,k,j=1}^s \{ & (\tau^{m_t^{kj}} A) \cap (\tau^{m_v^{rt}} A) + (\tau^{m_t^{kj}} A) \cap (\tau^{-n_v^{rt}} A) \\ & + (\tau^{-n_t^{kj}} A) \cap (\tau^{m_v^{rt}} A) + (\tau^{-n_t^{kj}} A) \cap (\tau^{-n_v^{rt}} A) \} + E_{v_l} \end{aligned}$$

where $\mu(E_{v_l}) \rightarrow 0$ as $v, l \rightarrow \infty$. From this and the strong mixing of τ it follows that $\mu(A) \leq s^4 [\mu(A)]^2$ for all $A \in \mathcal{F}$, which is impossible.

This theorem implies that τ^s has an n th root if and only if n divides s . Therefore we have the following result.

COROLLARY. *There exists a strongly mixing automorphism that has roots in a finitely generated class of integers and not in the complement of the class.*

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