ON THE EQUIVALENCE OF GAUSSIAN PROCESSES WITH FACTORABLE COVARIANCE FUNCTIONS

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ABSTRACT. Let $\{X_t\}_{t \in T}$ be a Gaussian process on a probability space $(\Omega, \mathcal{F}, P)$ with a factorable covariance function. We assume here that T is a p -dimensional Euclidean space. The purpose of this paper is to give necessary and sufficient conditions that a probability measure Q with respect to which $\{X_t\}_{t \in T}$ is a Gaussian process is equivalent to a probability measure P .

1. **Introduction.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let T be the p -dimensional Euclidean space and $(T, \mathcal{B}, \mu)$ be the Lebesgue measure space. We denote by $Z(E, \omega)$ a Gaussian random measure on $(T, \mathcal{B}, \mu)$, i.e., it is a Gaussian set function satisfying

$$(1.1) \quad E[Z(E, \omega)] = \int_{\Omega} Z(E, \omega) dP(\omega) = 0, \quad E \in \mathcal{B},$$

$$(1.2) \quad E[Z(E, \omega) \cdot Z(E', \omega)] = \mu(E \cap E')$$

for any E and E' in $\mathcal{B}^* = \{E \in \mathcal{B}; \mu(E) < \infty\}$ (for its definition see Itô [4]).

Let $L^2(T)$ denote the Hilbert space of square integrable functions with respect to μ . We shall consider a stochastic process $\{X_t\}_{t \in T}$ on $(\Omega, \mathcal{F}, P)$ which can be represented as

$$(1.3) \quad X_t = \int_T \varphi(u, t) dZ(u, \omega)$$

where $\varphi(\cdot, t) \in L^2(T)$ for each $t \in T$ and $\{\varphi(\cdot, t); t \in T\}$ spans $L^2(T)$. The process $\{X_t\}_{t \in T}$ so defined has the mean function identically equal to zero and its covariance function R is clearly given by

$$(1.4) \quad R(s, t) = E[X_s \cdot X_t] = \int_T \varphi(u, s) \varphi(u, t) d\mu(u).$$

Conversely a Gaussian process $\{X_t\}_{t \in T}$ with zero mean function and the covariance function given by (1.4) can be represented in the form of (1.3) (see Cramer [3]).

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The covariance function of the form (1.4) is called factorable (see Varberg [9]). The examples of Gaussian processes with factorable covariance functions are given in §3.

Let $\mathcal{F}_X$ be the σ -field generated by the process $\{X_t\}_{t \in T}$ and let Q be a probability measure under which the process $\{X_t\}_{t \in T}$ is Gaussian with the mean function $m(t)$ and the covariance function Γ_Q . It is not necessarily required that Γ_Q is factorable. We say that two probability measures Q and P are equivalent (designated as $Q \equiv P$) if they are mutually absolutely continuous. Our main result is the following theorem.

THEOREM. $Q \equiv P$ relative to $\mathcal{F}_X$ if and only if there is a symmetric kernel $K \in L^2(T \times T)$ such that

$$(1.5) \quad \Gamma_Q(s, t) = R(s, t) - \int_{T \times T} \varphi(u, s) \varphi(v, t) K(u, v) d\mu(u) d\mu(v),$$

$$(1.6) \quad 1 \notin \sigma(K),$$

and there is a function $k \in L^2(T)$ such that

$$(1.7) \quad m(t) = \int_T \varphi(u, t) k(u) d\mu(u),$$

where $\sigma(K)$ denotes the spectrum of the operator given by

$$(1.8) \quad (Kf)(u) = \int_T K(s, u) f(s) d\mu(s), \quad f \in L^2(T).$$

2. Proof of Theorem. We will first develop necessary terminologies. Let $H(R)$ denote the reproducing kernel Hilbert space with the reproducing kernel R and inner product $\langle \cdot, \cdot \rangle_{H(R)}$.

LEMMA 1. $H(R) \cong^J L^2(T)$ with $J(R(\cdot, t)) = \varphi(\cdot, t)$ and furthermore if $m \in H(R)$ and $J(m) = k$, then

$$m(t) = \int_T \varphi(u, t) k(u) d\mu(u).$$

Here $H_1 \cong^J H_2$ for two Hilbert spaces H_1 and H_2 means that J is a congruence between H_1 and H_2 , i.e. inner product preserving isomorphism from H_1 to H_2 .

The proof of the lemma is omitted since it can be easily shown by the similar method as the proof of Theorem 6 of Park [7] (only difference is to replace $1_u(\cdot)$ by $\varphi(\cdot, u)$).

Let $H(R) \otimes H(R)$ denote the tensor product of $H(R)$. Then it is again a reproducing kernel Hilbert space with the reproducing kernel $R(\cdot, t_1)R(\cdot, t_2)$. Let us denote the element of $H(R) \otimes H(R)$ by $f \otimes g$ where f and g are in $H(R)$. (See Aronszajn [1].)

If $h_1 \otimes h_2 \in H(R) \otimes H(R)$, then write

$$\sigma(h_1 \otimes h_2) = \frac{1}{2}[h_1 \otimes h_2 + h_2 \otimes h_1].$$

The closed linear subspace of $H(R) \otimes H(R)$ spanned by elements of the type $\sum_1^k c_n \sigma(h_1^n \otimes h_2^n)$ is called the symmetric tensor product of $H(R)$, denoted by $\sigma(H(R) \otimes H(R))$, and it has as elements all symmetric functions in $H(R) \otimes H(R)$. The following lemma is immediate.

LEMMA 2. $\sigma(H(R) \otimes H(R)) \cong J_1 \sigma(L^2(T) \otimes L^2(T))$ and if $F \in \sigma(H(R) \otimes H(R))$ and $J_1(F) = K$, then

$$F(s, t) = \int_{T \times T} \varphi(u, s) \varphi(v, t) K(u, v) d\mu(u) d\mu(v).$$

It is noted that if $K \in \sigma(L^2(T) \otimes L^2(T))$ then K is symmetric and $K \in L^2(T \times T)$. We will deduce our theorem using Theorem 1.3.1 of Oodaira [6], which states that $Q \equiv P$ relative to $\mathcal{F}_X$ if and only if

$$(2.1) \quad \Gamma_Q - R \in H(R) \otimes H(R),$$

(2.2) There are finite positive constants c_1 and c_2 such that $c_1 R \ll \Gamma_Q \ll c_2 R$, and

$$(2.3) \quad m \in H(R).$$

It is clear that the conditions (1.7) and (2.3) are identical by Lemma 1 and so are the conditions (1.5) and (2.1) by Lemma 2 (since $\Gamma_Q(\cdot, *) - R(\cdot, *) \in \sigma(H(R) \otimes H(R))$).

Let $\{\lambda_j\}$ and $\{g_j\}$ be eigenvalues and eigenfunctions of the operator K . Denote

$$G_j(t) = \int_T \varphi(u, t) g_j(u) d\mu(u).$$

Then $G_j \in H(R)$ and it corresponds to $g_j \in L^2(T)$ under the congruence. Furthermore

$$(2.4) \quad \begin{aligned} \langle G_j, G_j \rangle_{H(R)} &= (g_j, g_j) = 1, \\ \langle R(\cdot, t), G_j \rangle_{H(R)} &= G_j(t) \quad \text{and} \\ \langle \Gamma_Q(\cdot, t), G_j \rangle_{H(R)} &= (\varphi(*, t) - (K\varphi(\cdot, t))(*), g_j) = (1 - \lambda_j) G_j(t). \end{aligned}$$

The last identity follows from the fact that, under the condition (1.5),

$$\begin{aligned} \Gamma_Q(\cdot, t) &= \int_T \varphi(u, \cdot) \varphi(u, t) d\mu(u) - \int_T \left[\int_T \varphi(u, \cdot) \varphi(v, t) K(u, v) d\mu(v) \right] d\mu(u) \\ &= \int_T \varphi(u, \cdot) \left[\varphi(u, t) - \int_T \varphi(v, t) K(u, v) d\mu(v) \right] d\mu(u). \end{aligned}$$

Hence $\Gamma_Q(\cdot, t) \in H(R)$ for every $t \in T$ and it corresponds to $\varphi(\cdot, t) - (K\varphi(*, t))(\cdot)$ in $L^2(T)$ under the congruence.

Now $c_1 R \ll \Gamma_Q \ll c_2 R$ if and only if, according to (2.4), $c_1 G_j \leq (1 - \lambda_j) G_j \leq c_2 G_j$ for $j=1, 2, \dots$ if and only if $c_1 \leq 1 - \lambda_j \leq c_2$ for all j , i.e. $1 - c_2 \leq \lambda_j \leq 1 - c_1 < 1$, i.e. $1 \notin \sigma(K)$. Therefore the conditions (2.2) and (1.6) are identical. This completes the proof.

3. Examples of Gaussian processes with factorable covariance functions.
For the case that T is the real line.

(A) Brownian motion:

$$\begin{aligned} R(s, t) &= \min(s, t), \\ \varphi(s, t) &= 1_t(s) \text{ where } 1_t(s) = 1 \quad \text{if } s \leq t, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

(B) Gaussian Markov process:

$$\begin{aligned} R(s, t) &= u(s)v(t) \quad \text{if } s \leq t, \\ &= u(t)v(s) \quad \text{if } t < s, \\ \varphi(s, t) &= v(s) \left[\frac{d}{dt} \frac{u(t)}{v(t)} \right]^{1/2} \quad \text{if } s > t \text{ or } s = t = 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

(See Varberg [9] for the derivation of φ , and Beekman [2] for the examples of the process.)

For the case that T is p -dimensional Euclidean space E^p .

(C) p -parameter Gaussian process:

$$\begin{aligned} R(s, t) &= \prod_{j=1}^p \min(s_j, t_j), \\ \varphi(s, t) &= \prod_{j=1}^p 1_{t_j}(s_j) \quad \text{for } s = (s_1, \dots, s_p), t = (t_1, \dots, t_p). \end{aligned}$$

(See Park [7].)

(D) Brownian motion in Levy's sense: (p is odd)

$$R(s, t) = \frac{1}{2}(|s| + |t| - |s - t|)$$

where $|t| = (t_1^2 + \dots + t_p^2)^{1/2}$,

$$\varphi(s, t) = \frac{1}{\sqrt{\Omega_{p+1}}} (e^{i(s, t)} - 1) |s|^{-(p+1)/2}$$

where Ω_{p+1} is the area of the unit sphere in E^{p+1} . (See Molchan [5].) We remark that a proper modification of the theorem for the complex valued function $\varphi(s, t)$ can be easily seen.

For the case (A) and (C) the result is shown by Shepp [8] and Park [7] respectively. For the other case the theorem stated in this form is new as

far as we know. It is also remarked that if $Q \equiv P$, then the Radon-Nikodym derivative dQ/dP can be evaluated by a similar approach as Park [7], using the multiple Wiener integrals.

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