A CHARACTERIZATION OF p -REDUCTIVE LIE ALGEBRAS¹

ROBERT LEE WILSON

ABSTRACT. Let $\mathfrak{L}$ be a finite dimensional restricted Lie algebra over a field of characteristic $p > 0$. Then $\mathfrak{L}$ has a completely reducible faithful finite dimensional restricted representation if and only if $\mathfrak{L}$ has no nonzero nil ideals.

The finite dimensional Lie algebras possessing completely reducible faithful finite dimensional representations have been characterized by Jacobson, both over fields of characteristic zero [2] and over fields of prime characteristic [3]. In [4, pp. 98, 99] the similar question is raised of characterizing the finite dimensional restricted Lie algebras over a field of characteristic $p > 0$ possessing completely reducible faithful finite dimensional restricted representations. We call such Lie algebras p -reductive. A characterization of these algebras is given by:

THEOREM. *Let $\mathfrak{L}$ be a finite dimensional restricted Lie algebra over a field of characteristic $p > 0$. Then $\mathfrak{L}$ is p -reductive if and only if $\mathfrak{L}$ contains no nonzero nil ideals.*

(A subset $S \subseteq \mathfrak{L}$ is said to be *nil* if for every $x \in S$ there is some positive integer $n(x)$ such that $x^{p^{n(x)}} = 0$.)

PROOF. Let $\mathfrak{U}$ be the restricted enveloping algebra of $\mathfrak{L}$ and $\mathfrak{R}$ be the radical of $\mathfrak{U}$. Since $\mathfrak{L}$ generates $\mathfrak{U}$ the map $\Gamma \rightarrow \Gamma|_{\mathfrak{L}}$ is a one-one map of the representations of $\mathfrak{U}$ onto the representations of $\mathfrak{L}$ and Γ is completely reducible if and only if $\Gamma|_{\mathfrak{L}}$ is. Since a representation Γ of $\mathfrak{U}$ is completely reducible if and only if $\ker \Gamma \supseteq \mathfrak{R}$ we see that $\mathfrak{L}$ is p -reductive if and only if $\mathfrak{R} \cap \mathfrak{L} = (0)$. Since $\mathfrak{R} \cap \mathfrak{L}$ is clearly a nil ideal of $\mathfrak{L}$ which contains every nil ideal, the theorem is proved.

Let $\mathfrak{S}$ denote the solvable radical of $\mathfrak{L}$. Since every nil ideal of $\mathfrak{L}$ is contained in $\mathfrak{S}$ we have:

COROLLARY. *If $\mathfrak{S}$ is p -reductive so is $\mathfrak{L}$.*

Presented to the Society, January 21, 1971; received by the editors May 11, 1971.

AMS 1970 subject classifications. Primary 17B20.

Key words and phrases. Completely reducible restricted representations.

¹ The author has received partial support from a Sloan Foundation grant.

REMARKS. (1) The converse of the Corollary is not true. For example, if $\mathfrak{L}$ is any finite dimensional semisimple restricted Lie algebra and $\mathfrak{M}$ is an irreducible restricted $\mathfrak{L}$ module with basis $\{m_i | 1 \leq i \leq r\}$, $r \geq 2$, we may give the semidirect sum $\mathfrak{L} + \mathfrak{M}$ the structure of a restricted Lie algebra by setting $m_i^p = m_1$ for all i , $1 \leq i \leq r$. Then $\mathfrak{M}$ is the solvable radical of $\mathfrak{L} + \mathfrak{M}$, $\mathfrak{M}$ is not p -reductive (for $m_2 - m_1$ spans a nil ideal), and $\mathfrak{L} + \mathfrak{M}$ is p -reductive.

(2) Hochschild has shown [1] that every faithful finite dimensional restricted representation of $\mathfrak{L}$ is completely reducible if and only if $\mathfrak{L}$ is abelian and the map $x \rightarrow x^p$ is one-one. Thus every nonabelian p -reductive Lie algebra has a faithful finite dimensional restricted representation which is not completely reducible.

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903