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# AN ALTERNATIVE CONSTRUCTION OF $\beta X$ AND $vX$

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ABSTRACT. A different construction of $\beta X$ and $vX$ is given emphasizing the extension property and the role played by $C(X)$.

1. **Introduction.** That the existence of the Stone-Čech compactification of a completely regular space requires the axiom of choice is well known. In particular, it is equivalent to the Tychonoff product theorem for compact Hausdorff spaces [3, Chapters 6 and 11]. The definition of compactness causes this dependence. In [1] Comfort argues persuasively for an alternative definition (at least when dealing with rings of continuous real-valued functions) which, in the presence of the axiom of choice, is equivalent to the usual covering definition. A space $X$ is *compact** if every maximal ideal in $C^*(X)$ is fixed. He then obtains $\beta X$ by the usual method of embedding $X$ in a product of intervals and proves it is compact* without using the axiom of choice.

Our purpose here is to give an entirely different construction of $\beta X$, avoiding product spaces completely. This method emphasizes the extension property of $\beta X$ and is obtained directly from $C(X)$. Compact* is the more natural concept to use in this construction. There is also an analogous construction of $vX$. Notation follows [3]. All spaces are completely regular.

2. **Construction of $\beta X$.** A *$C^*$-extension of $X$* is a pair $(Y, r)$ where $r$ is an embedding of $X$ into $Y$ and $r(X)$ is a dense, $C^*$-embedded subspace of $Y$. Two $C^*$-extensions, $(Y_1, r_1)$ and $(Y_2, r_2)$, of $X$ are *equivalent* if there is a homeomorphism $h: Y_1 \to Y_2$ such that $h \circ r_1 = r_2$.

One can prove (without the axiom of choice [3, p. 136]) that if $(Y, r)$ is a $C^*$-extension of $X$ then $|Y| \leq |\mathscr{P}^2(X)|$, where $|Y|$ is the cardinal number of $Y$ and $\mathscr{P}^2(X) = \mathscr{P}(\mathscr{P}(X))$. Thus, there is a 1-1 function $\phi: Y \to \mathscr{P}^2(X)$. Define a topology on $\mathscr{P}^2(X)$ by taking as a base all subsets of $\mathscr{P}^2(X) \setminus \phi(Y)$ and all $U \subseteq \phi(Y)$ for which $\phi^{-1}(U)$ is open in $Y$. Then $\phi$ becomes an embedding and we can consider $Y$ to be a subspace of $\mathscr{P}^2(X)$ with this

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topology. Let $\mathscr{T}$ be the set of (distinct) Hausdorff topologies on $\mathscr{P}^2(X)$; let $E_Z=\{Z\}\times C(X,Z)$, where $Z$ is a subspace of $(\mathscr{P}^2(X),T)$, $T\in\mathscr{T}$; let $\mathscr{E}_1=\bigcup E_Z$ where the (disjoint) union is taken over all possible subspaces of $\mathscr{P}^2(X)$ with topologies from $\mathscr{T}$; and let $\mathscr{E}$ be the subspace of $\mathscr{E}_1$ consisting of those pairs $(Z,s)$ which are $C^*$-extensions of $X$. The following is now obvious.

LEMMA. *$\mathscr{E}$ is a set and if $(Y,r)$ is a $C^*$-extension of $X$ then $(Y,r)$ is equivalent to a member of $\mathscr{E}$.*

Let $\Lambda$ be the disjoint union of all $Y$ appearing as first element in a member of $\mathscr{E}$. On $\Lambda$ define an equivalence relation $\sim$ by saying $y_\alpha\sim y_{\alpha'}$ provided $f_\alpha(y_\alpha)=f_{\alpha'}(y_{\alpha'})$ for all $f\in C^*(X)$, where $y_\alpha\in Y_\alpha$, $y_{\alpha'}\in Y_{\alpha'}$, $(Y_\alpha,r_\alpha)$, $(Y_{\alpha'},r_{\alpha'})$ are members of $\mathscr{E}$ and $f_\alpha,f_{\alpha'}$ are the "extensions" of $f$ to $Y_\alpha$, $Y_{\alpha'}$: $f_\alpha\circ r_\alpha=f=f_{\alpha'}\circ r_{\alpha'}$. Let $\beta X$ be the set of equivalence classes of $\Lambda$ under $\sim$. To each $f\in C^*(X)$ there naturally corresponds a unique $f_\beta:\beta X\to R$ defined by $f_\beta([y_\alpha])=f_\alpha(y_\alpha)$. The topology on $\beta X$ is the weakest which makes all $f_\beta$ continuous.

THEOREM 1. *$\beta X$ is completely regular.*

PROOF. In view of [3, Theorem 3.7] all we must show is that $\beta X$ is Hausdorff. This is immediate from the fact that if $[y_\alpha]\neq[y_{\alpha'}]$ then there is an $f\in C^*(X)$ for which $f_\beta([y_\alpha])\neq f_\beta([y_{\alpha'}])$.

If we define $\beta:X\to\beta X$ by $\beta(x)=[r_\alpha(x)]$ where $r_\alpha$ is the second element of any member of $\mathscr{E}$, it is routine to verify that $(\beta X,\beta)$ is a $C^*$-extension of $X$. In particular, continuity follows since $f_\beta\circ\beta=f$; 1-1 follows since for $x_1\neq x_2$ we must have $r_\alpha(x_1)\nsim r_\alpha(x_2)$; openness follows from complete regularity of $X$.

For any extension $(Y_\alpha,r_\alpha)\in\mathscr{E}$ we define the canonical mapping $\eta_\alpha: Y_\alpha\to\beta X$ by $\eta_\alpha(y_\alpha)=[y_\alpha]$.

THEOREM 2. *For $(Y_\alpha,r_\alpha)\in\mathscr{E}$ we have $\eta_\alpha$ is continuous and for $f\in C^*(X)$, $f_\beta\circ\eta_\alpha=f_\alpha$.*

PROOF. Obvious.

THEOREM 3. *If $g:X\to Y$ is continuous and $Y$ is a compact Hausdorff space then there is a map $g_\beta:\beta X\to Y$ such that $g_\beta\circ\beta=g$.*

PROOF. [3, Theorem 6.4].

THEOREM 4. *$\beta X$ is compact*.*

PROOF. Let $M$ be a free maximal ideal in $C^*(\beta X)$ and let $p$ be a point not in $\beta X$. For each $f\in C^*(X)$ define $\tilde{f}: Y\to R$, where $Y=\beta X\cup\{p\}$, as follows.

$$\begin{aligned}\tilde{f}(y) &= f_\beta(y) \quad \text{if } y\in \beta X,\\ &= r_f \qquad \text{if } y = p,\end{aligned}$$

where $r_f$ is the real number corresponding to $f_\beta+M$ under the order isomorphism $C^*(\beta X)/M\cong R$. (See [**1**, Theorem 2.4*] or [**3**, Theorem 5.8] for details.) Give $Y$ the weakest topology which makes each element of $\{\tilde{f}\mid f\in C^*(X)\}$ continuous. Define $r:X\to Y$ by $r(x)=\beta(x)$. It is straightforward to verify that $(Y, r)$ is a $C^*$-extension of $X$. Thus, $(Y, r)$ is equivalent to some $(Y_1, r_1)\in\mathscr{E}$ and there is a homeomorphism $h: Y\to Y_1$ such that $h\circ r=r_1$. For each $f_\beta\in M$ we have $f_\beta(\eta_1[h(p)])=f_1[h(p)]=\tilde{f}(p)=0$, and we see that $M$ is fixed, a contradiction.

With the axiom of choice, compact* implies compact [**3**, Theorem 4.11] so that by [**3**, Theorem 6.5] (in view of Theorems 3 and 4) we have that $\beta X$, as we have constructed it, is the Stone-Čech compactification of $X$.

3. **Construction of** $vX$. We may construct $vX$ in precisely the same manner that we constructed $\beta X$ above, using $C(X)$ throughout in place of $C^*(X)$. Of the several possible definitions of realcompactness, we use the one in Engelking [**2**, p. 151] as most suitable here (in view of the emphasis on the extension aspect). The result analogous to Theorem 3, that if $g:X\to Y$ is continuous where $Y$ is realcompact then there is a map $g_v:vX\to Y$ such that $g_v\circ v=g$, follows from [**3**, Theorem 8.6]. The result analogous to Theorem 4 requires a different proof. Notice the fact that the axiom of choice is not used.

THEOREM 4'. *$vX$ is realcompact.*

PROOF. If not, then there is a space $Y$ and a homeomorphism $r:vX\to Y$ such that $r(vX)\neq(r(vX))^{-}=Y$ and for each $f_v\in C(vX)$ there is a $\tilde{f}\in C(Y)$ with $\tilde{f}\circ r=f_v$. Then $(Y, r\circ v)$ is a $C$-extension of $X$ and there is a continuous map $\eta: Y\to vX$ such that $\eta\circ r\circ v=v$. Thus, on the dense subspace $v(X)$ of $vX$, $\eta\circ r$ is the identity. We conclude $\eta\circ r$ is the identity mapping on all of $vX$. Similarly, $r\circ\eta$ is the identity on $Y$. Thus, $r(vX)=Y$, a contradiction. We conclude that $vX$ is realcompact.

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