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REPRESENTATIONS OF STRONGLY AMENABLE C^*-ALGEBRAS

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ABSTRACT. B. E. Johnson has introduced the concept of a strongly amenable C^*-algebra and has proved that GCR algebras and uniformly hyperfinite algebras are strongly amenable. We generalize the well-known Dixmier-Mackey theorem on amenable groups by proving that every continuous representation of a strongly amenable C^*-algebra is similar to a $*$-representation. As an application, we show that every invariant operator range for a Type I von Neumann algebra comes from an operator in the commutant.

Introduction. Let A be a complex Banach algebra. Then a complex Banach space X is a Banach A-module if it is a two-sided A-module and there exists a positive real number k such that for all $a\in A$ and $x\in X$ we have

$$\|ax\| \leqq k\,\|a\|\,\|x\| \quad \text{and} \quad \|xa\| \leqq k\,\|x\|\,\|a\|.$$

If X is a Banach A-module, then the dual space X^* becomes a Banach A-module if we define for $a\in A, f\in X^*$ and $x\in X$,

$$(af)(x) = f(xa), \quad (fa)(x) = f(ax).$$

A derivation from A into X^* is a complex linear map D from A into X^* such that $D(ab)=aD(b)+D(a)b$ for all, a, $b\in A$. If $f\in X^*$, the function $\delta(f)$ from A into X^* given by $\delta(f)(a)=af-fa$ is called the inner derivation induced by f. We recall that a topological group G is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on G [5]. B. E. Johnson has proved [7, Theorem 2.5] that if G is a locally compact topological group, then G is an amenable group if and only if for all $L^1(G)$-modules X and derivations D of $L^1(G)$ into X^*, we have that D is the inner derivation induced by an element of X^*. Johnson then defined a Banach algebra A to be amenable if every derivation of A into X^* is inner for all Banach A-modules X [7, §5]. Let A be a C^*-algebra and let A_e be the C^*-algebra obtained by adjoining the identity e. Johnson then makes the following definition [7, §7].

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DEFINITION. The C^*-algebra A is *strongly amenable* if, whenever X is a Banach A-module and D is a derivation of A into X^*, there is a $f \in \text{co}\{D(u)u^* : u \in U(A_e)\}$ with $D = -\delta(f)$, where X is made into a unital A_e-module by defining $xe = ex = x$ for all $x \in X$, D is extended to A_e by defining $D(e) = 0$, $U(A_e)$ is the unitary group of A_e, and co S denotes the w^*-closed convex hull of a set S contained in X^*.

If A has an identity, A is strongly amenable if and only if the definition is satisfied for all unital A-modules X with A_e replaced throughout by A [**7**, Proposition 7.2]. Johnson proved that the class of strongly amenable C^*-algebras contains all GCR C^*-algebras, and all uniformly hyperfinite C^*-algebras [**7**, Theorem 7.9 and Proposition 7.6]. He also proved [**7**, Proposition 7.8] that if G is an amenable locally compact group, then the C^*-group algebra [**2**, 13.9] is strongly amenable.

It is a well-known theorem, due to Dixmier [**1**], that every uniformly bounded strongly continuous representation of an amenable group on a Hilbert space is similar to a unitary representation. The main result of this paper is that every continuous representation of a strongly amenable C^*-algebra on a Hilbert space is similar to a *-representation.

The main results.

LEMMA 1. *Let A be a strongly amenable C^*-algebra, X a Banach A-module, and let $C = \{f \in X^* : af = fa$ for all $a \in A\}$. Then for all $f \in X^*$, $C \cap \text{co}\{ufu^* : u \in U(A_e)\}$ is nonempty.*

PROOF. The proof is a generalization of a proof of Johnson's [**7**, just before Proposition 7.14]. Let f be in X^* and let $\delta(f)$ be the inner derivation defined by f. Then there is a $g \in \text{co}\{\delta(f)(u)u^* : u \in U(A_e)\}$ such that $\delta(f) = -\delta(g)$. But $\delta(f)(u)u^* = ufu^* - f$, so $f + g \in \text{co}\{ufu^* : u \in U(A_e)\}$. Also, $\delta(f)(a) = -\delta(g)(a)$ for all $a \in A$, so $af - fa = ga - ag$, or $a(f+g) = (f+g)a$. Thus $f + g \in C$.

We recall that a linear functional f on a C^*-algebra A is called central if $f(ba) = f(ab)$ for all $a, b \in A$.

COROLLARY 1. *If A is strongly amenable C^*-algebra with identity, then A has positive central functionals of norm one and hence A has nonzero factor *-representations of finite type.*

PROOF. If f is any state of A, then by Lemma 1 (with $X = A$) there is a $g \in \text{co}\{ufu^* : u \in U(A)\}$ such that $g(ab) = g(ba)$ for all $a, b \in A$. Then g is clearly positive and $g(e) = 1$. The rest follows from [**2**, 6.8].

Since GCR algebras have only Type I *-representations and are strongly amenable, the following corollary is immediate.

COROLLARY 2. *If A is a GCR algebra with identity, then A has nonzero finite-dimensional *-representations.*

Let $B(H)$ be the bounded operators on Hilbert space H and let $K(H)$ be the compact operators on H.

COROLLARY 3. *The Calkin algebra $B(H)/K(H)$ (for H separable Hilbert space) is not strongly amenable, so $B(H)$ is not strongly amenable.*

PROOF. Let p be a projection in $B(H)$ with infinite-dimensional range and null-space, so that p is equivalent to the identity in $B(H)$. Then let $\bar{p}$ be the image of p in $B(H)/K(H)$. Since $B(H)/K(H)$ is simple [**10**, p. 291] it is clear that in any *-representation $\bar{p}$ will be a nontrivial projection equivalent to the identity. Thus every *-representation is infinite and Corollary 1 implies that $B(H)/K(H)$ is not strongly amenable. Since quotients of strongly amenable C^*-algebras are strongly amenable [**7**, 7.3], $B(H)$ is not strongly amenable.

For A a C^*-algebra, let $A\hat{\otimes}A$ be the completion of $A\otimes A$ in the greatest cross-norm. Then $(A\hat{\otimes}A)^*$ is the space of bounded bilinear functionals on $A\times A$ [**6**, p. 30]. We see that $A\hat{\otimes}A$ becomes a Banach A-module if we define for $a, b, c\in A$,

$$a(b\otimes c)=ab\otimes c, \qquad (b\otimes c)a=b\otimes ca.$$

Hence $(A\hat{\otimes}A)^*$ becomes a Banach A-module under the dual action: if $f\in(A\hat{\otimes}A)^*$ and $a, b, c\in A$,

$$(af)(b\otimes c)=f(b\otimes ca), \qquad (fa)(b\otimes c)=f(ab\otimes c).$$

Let $C=\{f\in(A\hat{\otimes}A)^*: af=fa \text{ for all } a\in A\}$.

We can also make $A\hat{\otimes}A$ and $(A\hat{\otimes}A)^*$ into Banach A-modules by defining for $f\in(A\hat{\otimes}A)^*$ and $a, b, c\in A$:

$$a\circ(b\otimes c)=b\otimes ac, \qquad (b\otimes c)\circ a=ba\otimes c,$$
$$(a\circ f)(b\otimes c)=f(ba\otimes c), \qquad (f\circ a)(b\otimes c)=f(b\otimes ac).$$

The map T in the following proposition takes the place of the invariant mean which is present in amenable groups.

PROPOSITION 1. *Let A be a strongly amenable C^*-algebra with identity e. Then there exists a linear map $T:(A\hat{\otimes}A)^*\to C$ such that:*

(a) *$T(f)\in\text{co}\{ufu^*: u\in U(A)\}$ for all $f\in(A\hat{\otimes}A)^*$, and*

(b) *$T(a\circ f)=a\circ T(f)$ and $T(f\circ a)=T(f)\circ a$ for all $a\in A$ and $f\in(A\hat{\otimes}A)$.*

PROOF. Let $Y=(A\hat{\otimes}A)^*\hat{\otimes}(A\hat{\otimes}A)$ be made into an A-module with operations $(f\otimes t)a=f\otimes(ta)$, $a(f\otimes t)=f\otimes(at)$, for $f\in(A\hat{\otimes}A)^*$, $t\in(A\hat{\otimes}A)$, and $a\in A$. Let Z be the closed submodule of Y spanned by elements of the form $(a\circ f)\otimes t-f\otimes(t\circ a)$ and $(f\circ a)\otimes t-f\otimes(a\circ t)$. Define an element F of Y^* by $F(f\otimes t)=f(t)$, $f\in(A\hat{\otimes}A)^*$, $t\in(A\hat{\otimes}A)$. Then F is zero on Z, so if we let $X=Y/Z$, we can regard F as an element of X^*. Then apply Lemma 1 to get an element T_0 of X^* such that $aT_0=T_0a$ for all a in A, and $T_0\in \mathrm{co}\{uFu^*\colon u\in U(A)\}$. Then define a bounded endomorphism T of $(A\hat{\otimes}A)^*$ by $T(f)(t)=T_0((f\otimes t)^-)$, where $(f\otimes t)^-$ means the coset of $f\otimes t$ in X. Then clearly T maps into C and satisfies (b). An application of the strong separation theorem shows that (a) is satisfied.

We use the existence of the function T to prove our main result.

THEOREM 1. *Every continuous representation of a strongly amenable C^*-algebra A on a Hilbert space is similar to a *-representation.*

PROOF. If $V\colon A\to B(H)$ is a continuous representation of A on H (i.e., V is a continuous algebra homomorphism of A into the bounded operators on a Hilbert space H), then V may be extended to the algebra A_e by defining $V'(a,\lambda)=V(a)+\lambda$. Then V' is a continuous representation of A_e, and A_e is strongly amenable if A is strongly amenable [7, 7.3]. So it suffices to assume that A has an identity and $V(e)=e$. Let $x, y\in H$ and define $f_{x,y}\in(A\hat{\otimes}A)^*$ by

$$f_{x,y}(a\otimes b) = (V(a)x, V(b^*)y).$$

We then define a new inner product on H by $(x,y)_1=T(f_{x,y})(e\otimes e)$. Then $(x,y)_1$ is a bounded sesquilinear form on H, so there is a bounded operator $R\in B(H)$ such that $(Rx,y)=T(f_{x,y})(e\otimes e)$ for all $x,y\in H$. Now $T(f_{x,x})\in \mathrm{co}\{uf_{x,x}u^*\colon u\in U(A)\}$ and for $u\in U(A)$ we have

$$uf_{x,x}u^*(e\otimes e) = f_{x,x}(u^*\otimes u) = \|V(u^*)x\|^2.$$

Then $\|x\|^2=\|V(u)V(u^*)x\|^2\leqq\|V\|^2\|V(u^*)x\|^2\leqq\|V\|^4\|x\|^2$, so

$$\|V\|^{-2}\,\|x\|^2 \leqq \|V(u^*)x\|^2 \leqq \|V\|^2\,\|x\|^2$$

for all $u\in U(A)$ and all $x\in H$. Thus

$$\|V\|^{-2}\,\|x\|^2 \leqq T(f_{x,x})(e\otimes e) \leqq \|V\|^2\,\|x\|^2.$$

Hence R is a positive invertible operator. Let S be the (positive) square root of R, then $(Sx,Sy)=T(f_{x,y})(e\otimes e)$. We now show that the representation $V'(a)=SV(a)S^{-1}$ is a *-representation. Let $u\in U(A)$, $z\in H$, $a, b\in A$ and compute:

$$f_{V(u)z,V(u)z}(a\otimes b) = (V(a)V(u)z, V(b^*)V(u)z) = f_{z,z}(au\otimes u^*b),$$

so that $f_{V(u)z,V(u)z}=u\circ f_{z,z}\circ u^*$. Then we have

$$\begin{aligned}(V(u)z, V(u)z)_1 &= T(f_{V(u)z,V(u)z})(e\otimes e) = T(u\circ f_{z,y}\circ u^*)(e\otimes e)\\ &= T(f_{z,z})(u\otimes u^*) = (u^*T(f_{z,z})u)(e\otimes e)\\ &= T(f_{z,z})(e\otimes e) = (z,z)_1.\end{aligned}$$

Finally,

$$(V'(u)z, V'(u)z) = (SV(u)S^{-1}z, SV(u)S^{-1}z) = (V(u)S^{-1}z, V(u)S^{-1}z)_1$$
$$= (S^{-1}z, S^{-1}z)_1 = (z, z).$$

In the above computation we used the fact that T satisfies both parts of property (b) in Proposition 1, and we used the fact that T maps into C. So V' is *-preserving on the unitary elements of A, and since every element of A is a linear combination of four unitary elements, we have that V' is a *-representation.

Applications and remarks. J. Dixmier has asked the following question: Let $A \subset B(H)$ be a von Neumann algebra. Suppose $b \in B(H)$ is such that $b(H)$ is invariant for A'=the commutant of A. Then does there exist an operator $a \in A$ with $a(H)=b(H)$? Using Theorem 1 we can answer this question in the affirmative in a special case.

PROPOSITION 2. *Let $A \subset B(H)$ be a strongly amenable C^*-algebra. Let $b \in B(H)$ be such that $b(H)$ is invariant under A. Then there is an operator $a \in A'$ with $a(H)=b(H)$.*

PROOF. The proof is almost exactly the same as a proof of C. Foiaş [**4**, Lemma 8] which uses Dixmier's theorem for representations of amenable groups. Since $(bb^*)^{1/2}$ and b have the same range, we may assume that $b \geqq 0$. Now define a representation $V: A \to B(H)$ as follows: For $d \in A$, $d(b(H)) \subset b(H)$, so if $x \in H$, there is an unique vector $y \in \text{null}(b)^\perp$ such that $(db)(x)=by$. Let $V(d)x=y$. Then $V(d)$ is well defined, $V(d)$ is clearly linear, and $V(d)$ has closed graph. So $V(d) \in B(H)$. Also $V(d)(H) \subset \text{null}(b)^\perp = (b(H))^-$ and $db=bV(d)$. It is then easily seen that V is linear with closed graph and $V(cd)=V(c)V(d)$. Hence $V: A \to B(H)$ is a continuous representation, so by Theorem 1, there is a positive invertible $S \in B(H)$ such that $U(d)=SV(d)S^{-1}$ is a *-representation. Let $S^{-1}b=ua$ be the polar decomposition of $S^{-1}b$. Then $bS^{-1}=au^*$, so $b=au^*S$ and $b(H) \subset a(H)$. Also $bS^{-1}u=a$, so $a(H) \subset b(H)$ and $a(H)=b(H)$. We will thus be finished when we show that $a \in A'$. Let $d \in A$. Then $db=bV(d)=bS^{-1}U(d)S$, so $dbS^{-1}=bS^{-1}U(d)$ and $dau^*=au^*U(d)$, or $da=au^*U(d)u$ for all $d \in A$. So we also have $d^*a=au^*U(d^*)u$, and since U is a *-representation we have $d^*a=au^*V(d)^*u$, or $ad=u^*U(d)ua$. Then $a^2d=au^*U(d)ua=da^2$. So $a^2 \in A'$ and $a \in A'$.

The following corollary is then immediate.

COROLLARY 4. *Let $A \subset B(H)$ be a von Neumann algebra such that A' contains a weakly dense strongly amenable C^*-algebra B. Suppose $b \in B(H)$ is such that $b(H)$ is invariant for B. Then $b(H)$ is also invariant for A' and there is an $a \in A$ with $a(H)=b(H)$.*

By [11] every Type I von Neumann algebra on a separable Hilbert space contains a weakly dense GCR algebra. So since GCR algebras are strongly amenable, the above corollary covers the case when A is a Type I von Neumann algebra on a separable Hilbert space, as well as the case when A' is a hyperfinite von Neumann algebra.

We close by remarking that Ehrenpreis and Mautner [3] give an example of a GCR group G which has a uniformly bounded representation on a Hilbert space that is not similar to a unitary representation. Our Theorem 1 then implies that the induced representation of $L^1(G)$ is not continuous in the C^*-algebra norm. The author does not know of an example of a continuous representation of a C^*-algebra on a Hilbert space that is not similar to a *-representation.

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