COERCIVE INEQUALITIES FOR CERTAIN CLASSES OF BOUNDED REGIONS

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ABSTRACT. In this paper, sufficient conditions are given for the coerciveness of formally positive integrodifferential forms over complex-valued functions satisfying zero boundary conditions in certain bounded domains in R_2 ; the boundaries need not be smooth. This work extends results given by the author in Comm. Pure Appl. Math., November 1969.

In addition, sufficient conditions and partial necessity conditions are given for coercive-type inequalities involving differential operators in the Hölder norm; here the results hold for complex-valued functions with no boundary conditions; the regions are bounded subdomains of R_n having the cone property.

Introduction. In a previous paper [6] the author stated and proved two sufficient conditions for coerciveness of formally positive, constant-coefficient, homogeneous integrodifferential forms over complex-valued functions defined on sectors in the plane. The first part of this paper extends these results to the nonconstant nonhomogeneous case and certain types of bounded subdomains on R_2 .

In the second part of this paper, some coerciveness-type results are derived for the Hölder norm.

I. L^2 -inequalities.

1.0. DEFINITIONS. If Ω is a domain in R_2 , let $C_t^m(\Omega)$ be the class of all complex-valued functions defined and continuous on $\bar{\Omega}$ and m times continuously differentiable on Ω , and all of whose derivatives of order $< t$ vanish on $\partial\Omega$, the boundary of Ω .

Let Ω_1 be an open sector (which may be a half-plane, but not a half-line) in R_2 with vertex at x_0 , and let B_ε be the open ball of radius ε about x_0 . We will say that a domain Ω^* is of *special type* if it is of the form $\Omega_1 \cap B_\varepsilon$.

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Let Ω be a bounded domain in R_2 . We say that Ω is of class P^s if every point x_0 of $\partial\Omega$ has a neighborhood $\mathcal{N}$ such that the closure of $\mathcal{N} \cap \Omega$ can be mapped in a one-to-one way onto the closure of a domain Ω^* of special type, with $\bar{\mathcal{N}} \cap \partial\Omega$ mapped into $\partial\Omega_1$, and such that the map and its inverse are s times continuously differentiable. (This class contains all domains of class C^s , and also others such as polygons and certain domains with piecewise-smooth boundaries.)

If u is a complex-valued function defined on Ω , we set

$$|u|_0^2(\Omega) = \int_{\Omega} |u(x)|^2 dx; \quad |u|_j^2(\Omega) = \sum_{|\alpha|=j} |D^\alpha u|_0^2; \quad \|u\|_m^2(\Omega) = \sum_{j=0}^m |u|_j^2(\Omega).$$

1.1. Statement of theorems.

THEOREM 1. Let Ω be a domain of class P^2 , and let a, b, c, d, e, f be complex-valued functions defined on $\bar{\Omega}$ such that a, b, c are continuous and d, e, f are bounded.

If $\operatorname{Re} a\bar{c} - |b|^2 > 0$ for all x in $\bar{\Omega}$, then $\exists K > 0$ such that

$$\|u\|_2^2(\Omega) \leq K \left[\iint_{\Omega} |a D_x^2 u + 2b D_x D_y u + c D_y^2 u + d D_x u + e D_y u + f u|^2 dx dy + \|u\|_0^2(\Omega) \right]$$

for all $u \in C_1^2(\Omega)$.

THEOREM 2. Let Ω be a domain of class P^m , and let A_α^{kj} ($|\alpha| = \alpha_1 + \dots + \alpha_n \leq m; j = 1, \dots, N; k = 1, \dots, s$) be complex-valued functions defined on $\bar{\Omega}$. Let $h(A_\alpha^{kj})$ be the highest-order part of A_α^{kj} ; i.e.,

$$\begin{aligned} h(A_\alpha^{kj}) &= A_\alpha^{kj}, & |\alpha| &= m, \\ &= 0, & |\alpha| &< m. \end{aligned}$$

Assume A_α^{kj} is bounded and $h(A_\alpha^{kj})$ is continuous on $\bar{\Omega}$.

Suppose the $N \times s$ matrix $h(A_\alpha^{kj}) \xi^\alpha$ of homogeneous m th degree polynomials in $\xi = (\xi_1, \xi_2)$ has the following property: the $N \times N$ subdeterminants of $h(A_\alpha^{kj}(x)) \xi^\alpha$ have no common nonzero real root, and, for $x \in \partial\Omega$, no more than one common polynomial factor (modulo constant factors) of the form $a(x) \xi_1 + \xi_2$, where $\operatorname{Im} a(x) \neq 0$.

Then $K > 0$ such that

$$\|u\|_m^2 \leq K \left[\sum_{k=1}^s \int_{\Omega} |A_\alpha^{kj} D^\alpha u_j|^2 dx_1 dx_2 + \|u\|_0^2 \right]$$

for all vector functions $u = (u_1, \dots, u_N) \in C_1^m(\Omega)$.

1.2. Outline of proof. The proof of both theorems follows by more-or-less standard techniques from similar results for constant-coefficient

operators and sectors in the plane proved by the author [6]. The method is as follows:

(1) Let $x_0 \in \partial\Omega$. Then $\exists \varepsilon > 0$ such that the conclusions of both theorems hold for all functions $u \in C_1^2(\Omega)$ ($C_1^m(\Omega)$) with support in $\mathcal{N}_\varepsilon(x_0)$.

(2) Since $\partial\Omega$ is compact, it can be covered by a finite number of neighborhoods of the type given in (1) above; hence $\bar{\Omega}$ can be covered by these plus a compact subset S of Ω . The result then follows by using a finite partition of unity relative to this covering, and Gårding's inequality [4] for the set S .

In order to prove (1) above, we map

$$\overline{\mathcal{N}_\varepsilon(x_0)} \cap \bar{\Omega}$$

onto a domain of special type; this transforms the differential operators $A_\alpha^k D^\alpha$ into new operators. We then notice that the hypotheses of Theorems 1 and 2 on the highest-order part of these operators are preserved. Then we prove (1) for the transformed neighborhood by using [6] together with the continuity of the highest-order coefficients, the boundedness of the lower-order coefficients, and an interpolation lemma for L^2 (see, e.g., Agmon [1, §3]). Finally, (1) follows for the given region by the two-way boundedness of the transformation.

For a more detailed treatment of this type of patching technique, see, e.g., Agmon [1], Schechter [7] or de Figueiredo [3].

II. Hölder inequalities.

2.0. DEFINITIONS. If Ω is a domain in R_2 , we denote by $C_0^\infty(\bar{\Omega})$ the class of all complex-valued functions defined and continuous on $\bar{\Omega}$, infinitely differentiable on Ω , and of compact support in $\bar{\Omega}$.

If l is a nonnegative integer and $f \in C_0^\infty(\bar{\Omega})$, let

$$[f]_l = [f]_l^\Omega = \text{l.u.b.}_{x \in \Omega; |\alpha|=l} |D^\alpha f(x)|, \quad |f|_l = |f|_l^\Omega = \sum_{j=0}^l [f]_j.$$

For $0 < \gamma < 1$ and $f \in C_0^\infty(\bar{\Omega})$, let

$$[f]_{l+\gamma} = \text{l.u.b.}_{P, Q \in \Omega; P \neq Q, |\alpha|=l} \left(\frac{|D^\alpha f(P) - D^\alpha f(Q)|}{|P - Q|^\gamma} \right), \quad |f|_{l+\gamma} = |f|_l + [f]_{l+\gamma}.$$

Let $C^{l+\gamma}(\bar{\Omega})$ be the completion in the $|\cdot|_{l+\gamma}$ norm of functions in $C_0^\infty(\bar{\Omega})$.

If $f = (f_1, \dots, f_N)$ is a complex-valued vector function and $l = (l_1, \dots, l_N)$ is a vector whose components are nonnegative integers,

our definitions are as follows:

$$[f]_l = \sum_{j=1}^N [f_j]_{l_j}, \quad |f|_l = \sum_{j=1}^N \sum_{i=0}^{l_j} [f_j]_i.$$

For $0 < \gamma < 1$,

$$[f]_{l+\gamma} = \sum_{j=1}^N [f_j]_{l_j+\gamma}, \quad |f|_{l+\gamma} = |f|_l + [f]_{l+\gamma}.$$

Again let $C^{l+\gamma}(\bar{\Omega})$ be the completion under the $|\cdot|_{l+\gamma}$ norm of vector-valued functions in $C_0^\infty(\bar{\Omega})$.

If $r = (r_1, \dots, r_N)$, $l = (l_1, \dots, l_N)$ and $0 \leq r_i \leq l_i$ ($i = 1, \dots, N$), let $C_r^{l+\gamma}(\bar{\Omega})$ be the subclass of functions in $C^{l+\gamma}(\bar{\Omega})$ whose derivatives of order $< r$ vanish on $\partial\Omega$. ($r < r'$ means $r_i < r'_i$ for $i = 1, \dots, N$.) In particular, $C_0^{l+\gamma}(\bar{\Omega}) = C^{l+\gamma}(\bar{\Omega})$.

If $\Omega \subseteq R_n$ is an unbounded domain, we say it has the *cone property* if $\exists$ an infinite right-circular cone ζ such that each point of $\bar{\Omega}$ can be taken as the vertex of the cone, suitably translated, so that $\bar{\zeta} \subseteq \bar{\Omega}$.

If Ω is a bounded domain, we say that it has the cone property if $\exists$ a finite number of right-circular cones, of fixed height, aperture and orientation, such that each point of $\bar{\Omega}$ may be taken as the apex of one of these cones, suitably translated, so that the cone is contained in $\bar{\Omega}$.

2.1. Statement of main theorems.

THEOREM 3. Let Ω be a bounded domain with the cone property. Consider the s differential expressions $A_\alpha^{k_j}(x) D^\alpha u_j$ [$k = 1, \dots, s$], where the implied summation is over $\alpha: 0 \leq |\alpha| \leq m_{ij}$, $j = 1, \dots, N$. Let γ ($0 < \gamma < 1$) be fixed. Suppose these forms satisfy the following three conditions:

(2.1) the $\{A_\alpha^{k_j}(x)\}$ are uniformly Hölder-continuous of order γ , and bounded in $\bar{\Omega}$;

(2.2) the operators are uniformly collectively elliptic; i.e., $\exists C > 0$ such that

$$|\xi|^{2m_j} |\eta_j|^2 \leq C \sum_{k=1}^s h(A_\alpha^{k_i}(x) \xi^\alpha \eta_i) \overline{h(A_\beta^{k_j}(x) \xi^\beta \eta_j)}$$

for all real n -vectors ξ , all complex N -vectors η , and all $x \in \bar{\Omega}$;

(2.3) the matrix of leading terms $h(A_\alpha^{k_i}(x) \xi^\alpha)$ [$i = 1, \dots, N$; $k = 1, \dots, s$] has rank N for every complex n -vector $\xi \neq 0$ and all $x \in \partial\Omega$.

Then $\exists K > 0$ such that

$$(2.4) \quad |u|_{m+\gamma}^\Omega \leq K \left(\sum_{k=1}^s [A_\alpha^{k_i} D^\alpha u_i]_\gamma^\Omega + |u|_0^\Omega \right)$$

for all $u \in C^{m+\gamma}(\bar{\Omega})$.

The following is a partial converse of Theorem 3.

THEOREM 4. *Let Ω be a bounded domain with the cone property; suppose the $\{A_\alpha^{ki}\}$ are constant and consist only of highest-order terms: i.e., $A_\alpha^{ki}\xi^\alpha = h(A_\alpha^{ki}\xi^\alpha)$ [$k=1, \dots, s$; $i=1, \dots, N$]. Suppose also that each $m_i \geq 1$. If (2.4) holds for all $u \in C^{m+\gamma}(\bar{\Omega})$, then (2.3) is true. If this is so, then (2.1) and (2.2) also hold.*

2.2. Outline of proof. To prove Theorem 3, we proceed in the following standard manner:

- (i) proof for homogeneous, constant-coefficient forms,
- (ii) local result in the general case,
- (iii) most general result by patching: finite partition of unity and Hölder-norm analogue of Gårding's inequality.

An explicit statement of (i) is now given.

LEMMA 1. *Let Ω be an unbounded domain with the cone property. For each $i=1, \dots, N$, let $\{A_\alpha^{ki}\xi^\alpha: k=1, \dots, s\}$ be homogeneous polynomials of degree m_i which satisfy condition (2.3) of Theorem 3. Let γ ($0 < \gamma < 1$) be fixed. Then $\exists K > 0$ such that*

$$[u]_{m+\gamma} \leq K \sum_{k=1}^s [A_\alpha^{ki} D^\alpha u_i]_\gamma$$

for all $u \in C_0^\infty(\bar{\Omega})$.

The proof of Lemma 1 follows from the Sobolev Representation Formula and the Hölder-Korn-Lichtenstein-Giraud inequality (see, e.g., [2, p. 223]) in a manner analogous to the method of K. T. Smith [8], the only difference is the use of the HKLG inequality in the Hölder norm instead of the Calderón-Zygmund inequality in L^p .

The proof of Theorem 3 now follows in the same manner as was outlined in §1.

An explicit proof of Theorem 4 will now be given.

PROOF OF THEOREM 4 (PROOF BY CONTRADICTION). Suppose the matrix $A_\alpha^{ki}\xi^\alpha$ [$k=1, \dots, s$; $i=1, \dots, N$] has rank $< N$ for $\xi=z$ where $z=(z_1, \dots, z_n) \in \mathbb{C}^n$, $z \neq 0$. Then $\exists$ complex $\eta_1, \dots, \eta_N$, not all zero, such that

$$(2.24) \quad \sum_{i=1}^N A_\alpha^{ki} z^\alpha \eta_i = 0 \quad [k=1, \dots, s].$$

For $x \in \bar{\Omega}$, define the function $u^l = (u_1^l, \dots, u_N^l)$ as follows:

$$(2.25) \quad u_i^l(x) = \eta_i e^{l \cdot x} / l^{m_i} \quad [l=1, 2, 3, \dots; i=1, \dots, N].$$

Then, using (2.24),

$$(2.26) \quad A_\alpha^{ki} D^\alpha u_i^l(x) = A_\alpha^{ki} z^\alpha \eta_i e^{l \cdot x} = 0 \quad [k=1, \dots, s].$$

Hence

$$(2.27) \quad \sum_{k=1}^s [A_{\alpha}^{ki} D^{\alpha} u_i^l]_{\gamma} = \sum_{k=1}^s [0]_{\gamma} = 0 \quad \forall l = 1, 2, 3, \dots$$

Let

$$(2.28) \quad M_l = \text{l.u.b.}_{x \in \Omega} |e^{l z \cdot x}|; \quad M_l > 0 \forall l;$$

$$\zeta = \max\{|z_1|, \dots, |z_n|\}; \quad \zeta > 0.$$

From (2.25), (2.27) and (2.28),

$$(2.29) \quad \sum_{k=1}^s [A_{\alpha}^{ki} D^{\alpha} u_i^l]_{\gamma} + |u^l|_0 = \sum_{i=1}^N |\eta_i| M_l / l^{m_i}.$$

On the other hand, by (2.25) and (2.28),

$$(2.30) \quad |u^l|_{m+\gamma} \geq |u^l|_1 = \sum_{i=1}^N |\eta_i| M_l (1 + l\zeta) / l^{m_i}.$$

Combining (2.29) with (2.30), we obtain

$$(2.31) \quad \frac{|u^l|_{m+\gamma}}{\sum_{k=1}^s [A_{\alpha}^{ki} D^{\alpha} u_i^l]_{\gamma} + |u^l|_0} \geq \frac{|u^l|_1}{\sum_{k=1}^s [A_{\alpha}^{ki} D^{\alpha} u_i^l]_{\gamma} + |u^l|_0} = 1 + l\zeta,$$

and this quantity $\rightarrow \infty$ as $l \rightarrow \infty$. Hence (2.4) does not hold. $C!$

This completes the proof of Theorem 4.

Concluding remarks. A partial bibliography of results obtained on the coerciveness problem was given by the author [6]. In addition, a number of mathematicians have obtained results on related types of "a priori inequalities;" see, e.g., Nečas [5]. Some of the outstanding unsolved problems will now be enumerated.

(1) What are necessary and sufficient conditions for (2.4) of Theorem 3 to hold if u satisfies zero boundary conditions: $u = D_n u = \dots = D_n^{r-1} u$ on $\partial\Omega$, for $0 < r < m$? (§2 of this paper solves the case $r=0$.)

(2) What are necessary and sufficient conditions for coerciveness in the L_2 norm of formally positive forms over complex-valued functions defined on subsets of R_n and satisfying r zero-type boundary conditions, as in (1)? (§1 of this paper gives some sufficient conditions for the case $r=1, n=2$.)

(3) Can any results be obtained for integrodifferential forms which are not formally positive? I.e., can we find conditions on the coefficients $a_{\alpha\beta}^{ij}$ for the form

$$\operatorname{Re} \int_{\Omega} a_{\alpha\beta}^{ij}(x) D^{\alpha} u_i \overline{D^{\beta} u_j} dx$$

to be coercive over complex-valued functions u_i satisfying zero-type boundary conditions on a nonsmooth $\partial\Omega$?

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