H-STRUCTURES ON PRODUCTS

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ABSTRACT. This note outlines a technique for determining the set of homotopy classes of H -structures on a product of H -spaces. The technique is then applied to computing the number of H -structures on $SO(4)$.

1. Introduction. This note outlines a technique for determining the set of homotopy classes of H -structures on a product $X = X_1 \times X_2 \times \cdots \times X_n$ of H -spaces. This technique is then applied to computing the number of H -structures on $SO(4) = SO(3) \times S^3$.

The technique may be summarized in the following statements.

(1) The set of homotopy classes of H -structures on X is in one-to-one correspondence with the homotopy set $[X \wedge X; X]$.

(2) There are subcomplexes $* = L_0 \subset L_1 \subset \cdots \subset L_{2n} = X \wedge X$ such that the sequences $0 \rightarrow [L_r/L_{r-1}; X] \rightarrow [L_r; X] \rightarrow [L_{r-1}; X] \rightarrow 0$ are exact for $r = 1, 2, \dots, 2n$.

(3) $[L_r/L_{r-1}; X]$ is the direct product of the loops $[X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}; X]$ with $p, q \geq 1$ and $p + q = r$.

(4) For any pointed space L , $[L; X]$ is the direct product of the loops $[L; X_k]$ for $k = 1, 2, \dots, n$.

Statement (4) is completely routine and (1) is essentially Theorem 5.5A of [1]. Statements (2) and (3) are established in §2. Throughout this note all spaces should be assumed to be CW-complexes of finite type.

As a corollary to the statements (1)–(4) above, one sees that the number $\#[X \wedge X; X]$ of products on X is the product of the numbers

$$\#[X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}; X_k].$$

2. Main results. We assume that each of the spaces X_k ($k = 1, 2, \dots, n$) is a path-connected, pointed CW-complex of finite type and has at least one H -structure. As usual, $*$ will be used ambiguously to denote base-points. Let $X = X_1 \times \cdots \times X_n$ and let $\phi: X \times X \rightarrow X \wedge X$ be the usual identification map. Suppose p, q are integers with $1 \leq p, q \leq n$ and that $\alpha = (i_1, \dots, i_p, j_1, \dots, j_q)$ is a $(p+q)$ -tuple of integers with $1 \leq i_1 < i_2 < \cdots < i_p \leq n$

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and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. Such an α will be called *admissible*. Define $\#\alpha = p + q$ and let $K_\alpha \subset X \wedge X$ be the set of points $\phi(x_1, \dots, x_n, x'_1, \dots, x'_n)$ with $x_i = x_j = *$ except when $i = i_1, \dots, i_p$ or $j = j_1, \dots, j_q$. Let $L_r = \bigcup \{K_\alpha \mid \#\alpha = r\}$ for $r = 2, 3, \dots, 2n$ and let $L_1 = * = \phi(X \vee X)$. Note that $L_1 \subset L_2 \subset \cdots \subset L_{2n} = X \wedge X$. Let $M_\alpha = X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}$.

2.1 LEMMA (JAMES). *If $Y_1, \dots, Y_m$ are subcomplexes of Y and if $r_i: Y \rightarrow Y_i$ are retractions with $r_i(Y_j) \subset Y_j$ for all i and j , then $Y_1 \cup \cdots \cup Y_m$ is retractile in Y [3, Lemma 3.1].*

2.2 LEMMA. *If $r \leq s$, then L_r is retractile in L_s .*

PROOF. In view of [8, Proposition (3.7)] it suffices to show each L_r is retractile in $X \wedge X$. But the projections

$$X \times X \rightarrow X_{i_1} \times \cdots \times X_{i_p} \times X_{j_1} \times \cdots \times X_{j_q}$$

induce retractions $X \wedge X \rightarrow K_\alpha$ satisfying the hypothesis of James' Lemma.

2.3 THEOREM. *The sequences*

$$0 \rightarrow [L_r/L_{r-1}; X] \rightarrow [L_r; X] \rightarrow [L_{r-1}; X] \rightarrow 0$$

are exact for $r = 2, 3, \dots, 2n$.

This is an immediate consequence of O'Neill's work [8, Theorems (3.3) and (3.4)]. Note that if X is the space of loops on an H -space, then a result of Eckmann and Hilton [2, Theorem 2.2] implies that these sequences split.

2.4 THEOREM. *$[L_r/L_{r-1}; X]$ is the direct product of the loops $[M_\alpha; X]$ with $\#\alpha = r$.*

PROOF. It suffices to establish that $L_r/L_{r-1} = \bigvee \{M_\alpha \mid \#\alpha = r\}$. But if $\alpha \neq \beta$ are admissible r -triples, then $K_\alpha \cap K_\beta \subset L_{r-1}$. The result now follows from the observation that $K_\alpha/(K_\alpha \cap L_{r-1}) = M_\alpha$.

3. Application to $SO(4)$.

3.1 THEOREM. *The space $X = SO(4)$ has $2^{86} \cdot 3^{16}$ homotopy classes of H -structures.*

In the course of the proof it will be convenient to refer to the following table of values of $[\Sigma^k P^2; S^3]$.

TABLE 3.2

$k =$	2	3	4	5	6	7	8	9
$[\Sigma^k P^2; S^3] =$	(2)	(4)	(2) + (2)	(2) + (2)	(4)	(2)	(1)	(2)

The symbol (m) denotes the abelian group of integers modulo m , and $+$ denotes direct sums. The entries are readily deduced from the Appendix to Chapter 12 of Hilton's book [4].

PROOF OF 3.1. We have $X_1 = P^3 = SO(3)$, $X_2 = S^3$. Note that since $X \wedge X$ is simply-connected, $[X \wedge X; S^3]$ is isomorphic to $[X \wedge X; X_1]$ under the 2-fold covering projection $S^3 \rightarrow P^3$. Let

$$a = \#[X_1 \wedge X_2 \wedge X_1 \wedge X_2; S^3] = \#[S^6 \wedge P^3 \wedge P^3; S^3],$$

$$b_1 = \#[X_1 \wedge X_1 \wedge X_2; S^3] = \#[S^3 \wedge P^3 \wedge P^3; S^3],$$

$$b_2 = \#[X_1 \wedge X_2 \wedge X_2; S^3] = \#[S^6 \wedge P^3; S^3];$$

$$c_1 = \#[X_1 \wedge X_1; S^3],$$

$$c_2 = \#[X_1 \wedge X_2; S^3] = \#[S^3 \wedge P^3; S^3] \quad \text{and}$$

$$c_3 = \#[X_2 \wedge X_2; S^3] = \#[S^6; S^3];$$

then $\#[X \wedge X; X] = a^2 b_1^4 b_2^4 c_1^2 c_2^2 c_3^2$. The evaluation of these numbers makes repeated use of Naylor's theorem that $S^3 \wedge P^3 = (S^3 \wedge P^2) \vee S^6$ [6]. In addition, we will use Table 3.2 several times in each calculation.

The value of a .

$$S^6 \wedge P^3 \wedge P^3 = ((S^3 \wedge P^2) \vee S^6) \wedge ((S^3 \wedge P^2) \vee S^6).$$

Thus

$$a = \#[S^6 \wedge P^2 \wedge P^2; S^3] \cdot (\#[S^9 \wedge P^2; S^3])^2 \cdot \#[S^{12}; S^3].$$

The cofibration sequence of spaces

$$\begin{array}{ccccccc} S^1 & \longrightarrow & S^1 & \longrightarrow & P^2 & \longrightarrow & S^2 \longrightarrow S^2 \\ & & f & & & & f \end{array}$$

in which f ambiguously denotes maps of degree 2, induces the cofibration sequence

$$\begin{array}{ccccccc} \Sigma^7 P^2 & \longrightarrow & \Sigma^7 P^2 & \longrightarrow & S^6 \wedge P^2 \wedge P^2 & \longrightarrow & \Sigma^8 P^2 \longrightarrow \Sigma^8 P^2 \\ & & f & & & & f \end{array}$$

and hence the exact sequence of groups

$$\longleftarrow_{f^*} [\Sigma^7 P^2; S^3] \longleftarrow [S^6 \wedge P^2 \wedge P^2; S^3] \longleftarrow [\Sigma^8 P^2; S^3] \longleftarrow_{f^*}.$$

The two outer groups are repetitions and have been omitted from the line above. Since $[\Sigma^7 P^2; S^3] = \mathbb{Z}_2$ and $[\Sigma^8 P^2; S^3] = 0$, both homomorphisms f^* are trivial, and $\#[S^6 \wedge P^2 \wedge P^2; S^3] = 2$.

On the other hand, $\#[\Sigma^9 P^2; S^3] = 2$ and $\#[S^{12}; S^3] = 4$. Thus $a = 2^5$.

The value of b_1 .

$$\begin{aligned} S^3 \wedge P^3 \wedge P^3 &= ((S^3 \wedge P^2) \wedge S^6) \wedge P^3 = (P^2 \wedge S^3 \wedge P^3) \vee S^6 \wedge P^3 \\ &= (S^3 \wedge P^2 \wedge P^2) \vee (\Sigma^6 P^2) \vee (\Sigma^8 P^2) \vee S^9. \end{aligned}$$

Thus $b_1 = \#[S^3 \wedge P^2 \wedge P^2; S^3] \cdot (\#[\Sigma^6 P^2; S^3])^2 \cdot \#[S^9; S^3]$. As before, we have an exact sequence

$$\xleftarrow{f^*} [\Sigma^4 P^2; S^3] \xleftarrow{\quad} [S^3 \wedge P^2 \wedge P^2; S^3] \xleftarrow{\quad} [\Sigma^5 P^2; S^3] \xleftarrow{f^*}.$$

But $[\Sigma^4 P^2; S^3] \approx [\Sigma^5 P^2; S^3] \approx Z_2 + Z_2$, and f^* is trivial on each of these. This shows $\#[S^3 \wedge P^2 \wedge P^2; S^3] = 2^4$.

On the other hand $\#[\Sigma^6 P^2; S^3] = 2^2$ and $\#\pi_9(S^3) = 3$, whence $b_1 = 2^8 \cdot 3$.

The values of b_2 , c_1 , c_2 and c_3 .

$$b_2 = \#[S^6 \wedge P^3; S^3] = \#[S^6 \wedge P^2; S^3] \cdot \#[S^9; S^3] = 2^2 \cdot 3;$$

$$c_1 = \#[P^3 \wedge P^3; S^3] = 2^8 \cdot 3 \quad [4];$$

$$c_2 = \#[S^3 \wedge P^3; S^3] = \#[S^3 \wedge P^2; S^3] \cdot \#[S^6; S^3] = 4 \cdot 12 = 2^4 \cdot 3;$$

$$c_3 = \#\pi_6(S^3) = 2^2 \cdot 3.$$

Thus $\#[X \wedge X; X] = 2^{86} \cdot 3^{16}$.

A similar calculation would show that $S^3 \times S^3$ has $2^{20} \cdot 3^{16}$ homotopy classes of H -structures. This result was obtained earlier by Norman [7] using Loibel's Theorem [5].

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