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ON HIGH ORDER DERIVATIONS OF FIELDS

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ABSTRACT. Let $\mathscr{D}(L/K)$ denote the derivation algebra of a field extension L/K of prime characteristic. If L/K is purely inseparable and has an exponent, then every intermediate field F of L/K equals the center of $\mathscr{D}(L/F)$. Here we prove the converse of this statement.

Let $\mathscr{D}(L/K)=L\oplus\mathscr{D}_0(L/K)$ denote the derivation algebra of a field extension L/K where $\mathscr{D}_0(L/K)$ is the set of all high order derivations of L/K [**3**, pp. 1 and 6]. In [**4**, p. 19, Theorem 3], it is shown that if L/K is purely inseparable and has an exponent, then $F=Z(\mathscr{D}(L/F))$ for every intermediate field F of L/K where $Z(\mathscr{D}(L/F))$ denotes the center of $\mathscr{D}(L/F)$. This result permits a Galois correspondence between the intermediate fields of L/K and closed subrings of $\mathscr{D}(L/K)$ containing L. In this note, we show that the converse of this result is true; that is, for an arbitrary field extension L/K of characteristic $p>0$, if $F=Z(\mathscr{D}(L/F))$ for every intermediate field F of L/K, then L/K is purely inseparable and has an exponent.

Unless otherwise specified, L/K always denotes a nontrivial field extension of characteristic $p>0$.

Our notation coincides with that in [3] and [**4**]. The set of qth order derivations of L/K into L is denoted by $\mathscr{D}_0^{(q)}(L/K)$. Thus $\mathscr{D}_0(L/K)=\bigcup_{q=1}^{\infty}\mathscr{D}_0^{(q)}(L/K)$. $C_q(L/F)$ denotes the set, $\{x \mid x\in L$, for all $D\in\mathscr{D}_0^{(q)}(L/F)$, $D(x)=0\}$.

For any intermediate field F of L/K, $Z(\mathscr{D}(L/F))$ is an intermediate field of L/K containing F and equals the set $\{x \mid x\in L$, for all $D\in\mathscr{D}_0(L/F)$, $D(x)=0\}$.

THEOREM. *The following conditions are equivalent*:

(1) *For every intermediate field F of L/K, $F=Z(\mathscr{D}(L/F))$.*

(2) *For every intermediate field F of L/K, $F=\bigcap_{i=1}^{\infty}F(L^{p^i})$.*

(3) *For every intermediate field F of L/K, every relative p-base of F/K is a generating set of F/K.*

(4) *L/K is purely inseparable and has an exponent.*

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PROOF. (1)$\Leftrightarrow$(2). This equivalence is immediate once we show that for any F, $Z(\mathscr{D}(L/F))=\bigcap_{i=1}^{\infty} F(L^{p^i})$. Let $x\in\bigcap_{i=1}^{\infty} F(L^{p^i})$. Then, for any q and any $D\in\mathscr{D}_0^{(q)}(L/F)$, $D(x)=0$ since $F(L^{p^q})\subseteq C_q(L/F)$ by [**3**, p. 5, Corollary 7.1]. Since $\mathscr{D}_0(L/F)=\bigcup_{q=1}^{\infty}\mathscr{D}_0^{(q)}(L/F)$, we have $x\in Z(\mathscr{D}(L/F))$. Hence $\bigcap_{i=1}^{\infty} F(L^{p^i})\subseteq Z(\mathscr{D}(L/F))$. Let $x\in Z(\mathscr{D}(L/F))$. If $x\notin\bigcap_{i=1}^{\infty} F(L^{p^i})$, then there exists an i such that $x\notin F(L^{p^i})$. In this case, by [**4**, p. 18, Theorem 2], there exists $D\in\mathscr{D}_0(L/F(L^{p^i}))$ (whence $D\in\mathscr{D}_0(L/F)$) such that $D(x)\neq 0$ contrary to the fact that $x\in Z(\mathscr{D}(L/F))$. Thus $x\in\bigcap_{i=1}^{\infty} F(L^{p^i})$ so that $Z(\mathscr{D}(L/F))\subseteq \bigcap_{i=1}^{\infty} F(L^{p^i})$.

(3)$\Leftrightarrow$(4). Suppose (3) holds. Let F be any intermediate field of L/K such that $F\supset K$ (strict inclusion). Let M be any relative p-base of F/K. Since $F=K(M)$ and $F\supset K$, $M\neq\varnothing$. If F/K is separable, then M is algebraically independent over K. In this case, the relative p-base $M^{p+1}=\{m^{p+1}|m\in M\}$ of F/K cannot generate F/K else we contradict the algebraic independence of M over K. Thus F/K cannot be separable. Hence L/K has no intermediate fields (other than K) which are separable over K. Thus L/K is purely inseparable. That L/K has an exponent now follows by [**1**, p. 240, Corollary to Theorem 1] or [**2**, p. 12, Corollary 1.18]. By [**2**, p. 2, Corollary 1.6], we have that (4) implies (3).

(2)$\Rightarrow$(3). Suppose there exists an intermediate field L' of L/K for which there exists a relative p-base M such that $L'\supset K(M)$. Set $F=K(M)$. Then $L'=F(L'^{p^i})$, $i=1, 2, \cdots$. Thus $L'(L^{p^i})=F(L^{p^i})$, $i=1, 2, \cdots$, so that $\bigcap_{i=1}^{\infty} F(L^{p^i})=\bigcap_{i=1}^{\infty} L'(L^{p^i})\supseteq L'\supset F$, a direct contradiction of (2).

(4)$\Rightarrow$(2). Immediate. q.e.d.

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