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POSITIVE TRANSFORMATIONS RESTRICTED TO SUBSPACES AND INEQUALITIES AMONG THEIR PROPER VALUES

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ABSTRACT. Let A be a positive Hermitian transformation on an n-dimensional unitary space E_n with proper values $a_1 \geqq \cdots \geqq a_n$. Let $b_1 \geqq \cdots \geqq b_k$ be the proper values of $A|M$, where M is a proper subspace of E_n and $c_1 \geqq \cdots \geqq c_h$ be the proper values of $A|M^\perp$. Let $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$ be sequences of positive integers, with $i_r \leqq k$ and $j_r \leqq h$. Then $(b_{i_1} \cdots b_{i_r}) \cdot (c_{j_1} \cdots c_{j_r}) \geqq (a_{n-r+1} \cdots a_n)(a_{i_1+j_1-1} \cdots a_{i_r+j_r-1})$. In this article generalizations of this inequality have been studied.

Let A be a positive Hermitian linear transformation on a unitary space E_n with proper values $a_1 \geqq \cdots \geqq a_n$. Let M be a proper subspace of E_n. Let the proper values of $A|M$ be $b_1 \geqq \cdots \geqq b_h$ and the proper values of $A|M^\perp$ be $c_1 \geqq \cdots \geqq c_k$. Then N. Aronszajn **[4]** has given the inequality $a_{i+j-1} \leqq b_i + c_j$, for $1 \leqq i \leqq h$ and $1 \leqq j \leqq k$. Generalizations of this inequality have been given by A. J. Hoffman, R. C. Thompson and L. J. Freede **[5]**. All of these inequalities involve sums of proper values. In this article we shall give generalizations of these inequalities containing products of proper values.

1. **Definitions and notations.** The inner product of two vectors α and β will be denoted by (α, β). The determinant of a linear transformation A on E_n will be denoted by det A. The identity transformation will be denoted by I. A Hermitian linear transformation is called positive if $(A\xi, \xi) > 0$ for all $\xi \neq 0$. An orthonormal set $\{\alpha_1, \cdots, \alpha_k\}$ will be indicated by $\{\alpha_p\}$ o.n. The subspace spanned by the set $\{\alpha_1, \cdots, \alpha_k\}$ will be denoted by $[\alpha_1, \cdots, \alpha_k]$. We write dim $M = h$ if the dimension of the subspace M is h.

If A is a linear transformation on a unitary space E_n and if M is a subspace of E_n, then we define a linear transformation $A|M$ as follows: if $\xi \in M$, let $[A|M]\xi = PA\xi$, where P is the orthogonal projection on M. We observe that if α and $\beta \in M$, then $([A|M]\alpha, \beta) = (PA\alpha, \beta) = (A\alpha, \beta)$. It follows that if A is Hermitian (positive), then so is $A|M$.

If $j_p \leqq i_p$, for $p = 1, \cdots, k$, we write $(j_1, \cdots, j_k) \leqq (i_i, \cdots, i_k)$ and say the sequence $(i_1, \cdots, i_k)$ is greater than or equal to the sequence

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$(j_1, \cdots, j_k)$. Further, given any sequence $i_1 \leqq \cdots \leqq i_k$ of positive integers, we define $(i_1'', \cdots, i_k'')$ to be the strictly increasing sequence of positive integers such that $(i_1, \cdots, i_k) \leqq (i_1'', \cdots, i_k'')$ and $(i_1'', \cdots, i_k'') \leqq (j_1, \cdots, j_k)$, if $(j_1, \cdots, j_k)$ is a strictly increasing sequence of positive integers greater than or equal to $(i_1, \cdots, i_k)$ [**1**].

2. **Some theorems.** Let A be a positive transformation on E_n with proper values $m_1 \geqq \cdots \geqq m_n$. Then

$$(1) \qquad m_1 \cdots m_k = \sup_{\{\xi_i\} \text{o.n.}} \det((A\xi_i, \xi_j)).$$

This theorem is due to Ky Fan [**3**]. Further, if $i_1 \leqq \cdots \leqq i_k$ is a sequence of positive integers such that $i_p \leqq n-k+p$, for $p=1, \cdots, k$, and $k \leqq n$, then

$$(2) \qquad \inf_{M_1 \subset \cdots \subset M_k; a_p} \ \sup_{\substack{\xi_p \perp M_p \\ \{\xi_p\}\text{o.n.}}} \det((A\xi_i, \xi_j))_{1\leqq i \leqq k; 1 \leqq j \leqq k} = m_{i_1''} \cdots m_{i_k''},$$

where a_p stands for dim $M_p = i_p - 1$ and M_p is a subspace of E_n [**1**].

If A is a Hermitian linear transformation on E_n with proper values $p_1 \geqq \cdots \geqq p_n$ and $t_1 \geqq \cdots \geqq t_k$ are the proper values of $A|M$, where M is a subspace of E_n and dim $M = k$, then

$$(3) \qquad p_{n-k+i} \leqq t_i \leqq p_i,$$

for $i = 1, \cdots, k$ [**2**].

3. Theorem. *Let A be a positive transformation on E_n with proper values $a_1 \geqq \cdots \geqq a_n$. Let $R_1, \cdots, R_s$ be proper subspaces of E_n such that R_i is orthogonal to R_j, for $i \neq j$, $E_n = R_1 \oplus \cdots \oplus R_s$, and* dim $R_q = h_q$, *for $q = 1, \cdots, s$. Suppose the proper values of $A|R_q$ are $b_{q1} \geqq \cdots \geqq n_{qh_q}$, $q=1, \cdots, s$. Let $i_{q1} \leqq \cdots \leqq i_{qr}$, $q=1, \cdots, s$, be sequences of positive integers such that $i_{qp} \leqq h_q - r + p$, for $p = 1, \cdots, r$, $q = 1, \cdots, s$, with r less than or equal to the* $\min(h_1, \cdots, h_s)$. *Then*

$$(1) \qquad \prod_{q=1}^{s} \left\{ \prod_{p=1}^{r} b_{q.i_{qp}''} \right\} \geqq \left\{ \prod_{p=n-r(s-1)+1}^{n} a_p \right\} \left\{ \prod_{p=1}^{r} a_{v_p} \right\}$$

where $v_p = (1 - s + \sum_{q=1}^{s} i_{qp})''$.

Proof. By §2 (2) there exist subspaces $M_{q1} \subset \cdots \subset M_{qr} \subset R_q$, $q = 1, \cdots, s$, with dim $M_{qp} = i_{qp} - 1$, $p = 1, \cdots, r$, $q = 1, \cdots, s$, such that

$$\prod_{p=1}^{r} b_{q.i_{qp}''} = \sup_{\eta_{qp} \perp M_{qp}; \{\eta_{qp}\}\text{o.n.}} \det(([A \mid R_q]\eta_{qi}, \eta_{qj}))_{1 \leqq i \leqq r; 1 \leqq j \leqq r}$$

$$(2) \qquad = \sup_{\eta_{qp} \perp M_{qp}; \{\eta_{qp}\}\text{o.n.}} \det((A\eta_{qi}, \eta_{qj}))_{1 \leqq i \leqq r; 1 \leqq j \leqq r} \quad \text{for } q = 1, \cdots, s.$$

Let $L_p=M_{1p}\oplus\cdots\oplus M_{sp}$, $p=1,\cdots,r$. We observe that $L_1\subset\cdots\subset L_r\subset E_n$ and $\dim L_p=(1-s+\sum_{q=1}^{s} i_{qp})-1$, $p=1,\cdots,r$. Let $\{\zeta_1,\cdots,\zeta_r\}$ be an orthonormal set in E_n such that $\zeta_p\perp L_p, p=1,\cdots,r$. Now, for each $p=1,\cdots,r$, there exists an orthonormal set $\{\eta_{1p},\cdots,\eta_{sp}\}$ such that $\zeta_p\in[\eta_{1p},\cdots,\eta_{sp}]$ and $\eta_{qp}\in M_{qp}^{\perp}\cap R_q$, $q=1,\cdots,s$. It is clear that there exists an orthonormal set $\{\eta'_{11},\cdots,\eta'_{1r},\eta'_{21},\cdots,\eta'_{2r},\cdots,\eta'_{s1},\cdots,\eta'_{sr}\}$ such that $\eta'_{qp}\in M_{qp}^{\perp}\cap R_q$, $q=1,\cdots,s$, $p=1,\cdots,r$, with

$$[\eta_{11},\cdots,\eta_{1r},\cdots,\eta_{s1},\cdots,\eta_{sr}]\subset[\eta'_{11},\cdots,\eta'_{1r},\cdots,\eta'_{s1},\cdots,\eta'_{sr}].$$

We extend $\{\zeta_1,\cdots,\zeta_r\}$ to an orthonormal set $\{\zeta_1,\cdots,\zeta_{sr}\}$ in such a way that $L=[\eta'_{11},\cdots,\eta'_{1r},\cdots,\eta'_{s1},\cdots,\eta'_{sr}]=[\zeta_1,\cdots,\zeta_{sr}]$. Thus

$$\text{(3)}\quad \det\begin{pmatrix}(A\eta'_{11},\eta'_{11}) & \cdots & (A\eta'_{11},\eta'_{sr})\\ \cdots & & \cdots\\ (A\eta'_{sr},\eta'_{11}) & \cdots & (A\eta'_{sr},\eta'_{sr})\end{pmatrix} = \det(A\mid L)=\det((A\zeta_i,\zeta_j))_{1\leqq i\leqq sr;1\leqq j\leqq sr}.$$

Consequently

$$\text{(4)}\quad \prod_{q=1}^{s}\det((A\eta'_{qi},\eta'_{qj}))_{1\leqq i\leqq r;1\leqq j\leqq r}\geqq\det((A\zeta_i,\zeta_j))_{1\leqq i\leqq sr;1\leqq j\leqq sr}.$$

Suppose $d_1\geqq\cdots\geqq d_{sr}$ are the proper values of $A|L$. By §2 (1) we obtain

$$\text{(5)}\quad d_1\cdots d_r\geqq(([A\mid L]\zeta_i,\zeta_j))_{1\leqq i\leqq r;1\leqq j\leqq r}=\det((A\zeta_i,\zeta_j))_{1\leqq i\leqq r;1\leqq j\leqq r}.$$

By §2 (3), it follows that

$$\text{(6)}\quad d_{r+1}\cdots d_{sr}\geqq\prod_{p=n-r(s-1)+1}^{n}a_p.$$

Combining (4), (5) and (6) we obtain

$$\text{(7)}\quad \prod_{q=1}^{s}\det((A\eta'_{qi},\eta'_{qj}))_{1\leqq i\leqq r;1\leqq j\leqq r}\geqq\left(\prod_{p=n-r(s-1)+1}^{n}a_p\right)\det((A\zeta_i,\zeta_j))_{1\leqq i\leqq r;1\leqq j\leqq r}.$$

Using (2) and (7) we obtain

$$\text{(8)}\quad \prod_{q=1}^{s}\left\{\prod_{p=1}^{r}b_{q,i''_{qp}}\right\}\geqq\left\{\prod_{p=n-r(s-1)+1}^{n}a_p\right\}\left\{\inf_{K_1\subset\cdots\subset K_r;w_p}\ \sup_{\substack{\delta_p\perp k_p\\ \{\delta_p\}\text{o.n.}}}\det((A\delta_i,\delta_j))_{1\leqq i\leqq r;1\leqq j\leqq r}\right\}$$

where w_p stands for dim $K_p=(1-s+\sum_{q=1}^{s} i_{qp})-1$. But by §2 (2) we obtain

$$(9) \qquad \inf_{K_1\subset\cdots\subset K_r;w_p} \sup_{\substack{\delta_p\perp k_p \\ \{\delta_p\}\text{o.n.}}} \det((A\delta_i, \delta_j))_{1\leqq i\leqq r;1\leqq j\leqq r} = \prod_{p=1}^{r} a_{v_p}$$

where w_p stands for dim $K_p=(1-s+\sum_{q=1}^{s} i_{qp})-1$ and

$$v_p = \left(1 - s + \sum_{q=1}^{s} i_{qp}\right)''.$$

Combining (8) and (9) we obtain (1); thus the proof is complete.

Indeed this theorem is true for a nonnegative transformation on E_n.

4. COROLLARY. *Let H be a Hermitian transformation on E_n with proper values $a_1\geqq\cdots\geqq a_n$. Let a be any real number such that $a\leqq a_n$. Then it is clear that $H-aI$ is nonnegative. Let us consider subspaces and sequences of positive integers of* §3. *Let the proper values of $H|R_q$ be $b_{q1}\geqq\cdots\geqq b_{qh_q}$, $q=1,\cdots,s$. Then applying* §3 *to $H-aI$ we obtain*

$$\left\{\prod_{q=1}^{s}\left(\prod_{p=1}^{r}(b_{q,i''_{qp}} - a)\right)\right\} \geqq \left\{\prod_{p=n-r(s-1)+1}^{n}(a_p - a)\right\}\left\{\prod_{p=1}^{r}[a_{v_p} - a]\right\}$$

where $v_p=(1-s+\sum_{q=1}^{s} i_{qp})''$.

5. DEFINITION. Let $i_1\leqq\cdots\leqq i_k$ be a sequence of positive integers such that $i_p\geqq p$, for $p=1,\cdots,k$. We define $(i'_1,\cdots,i'_k)$ to be the strictly increasing sequence of positive integers such that $(i'_1,\cdots,i'_k)\leqq(i_1,\cdots,i_k)$ and $(j_1,\cdots,j_k)\leqq(i'_1,\cdots,i'_k)$, if $(j_1,\cdots,j_k)$ is a strictly increasing sequence of positive integers which is less than or equal to $(i_1,\cdots,i_k)$.

6. REMARK. For a Hermitian linear transformation R. C. Thompson and L. J. Freede [5] have shown that

$$\sum_{q=1}^{s}\left(\sum_{p=1}^{r} b_{q,i'_q}\right) \leqq \left(\sum_{p=1}^{r(s-1)} a_p\right) + \sum_{p=1}^{r} a_{x_p},$$

where $x_p=(\sum_{q=1}^{s} i_{qp})'$ and where the symbols are defined in §3 except the obvious changes are made in the conditions on the sequences $i_{q1},\cdots,i_{qr}$. Thus we might expect a similar inequality for products. But this conjecture is refuted by the following example. Let A be represented by

$$\begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is clear that A is positive. Consider the subdivision

$$\left(\begin{array}{c|c} 9 & 1 \\ \hline 1 & 1 \end{array}\right).$$

Then $b_{1,i''_{11}}b_{2,i''_{21}}=9>8=a_1a_2$.

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