PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

AN ABSTRACT MEASURE DIFFERENTIAL EQUATION

R. R. SHARMA

ABSTRACT. An abstract measure differential equation is introduced as a generalization of ordinary differential equations and measure differential equations. The existence and extension of solutions of this equation are considered.

Introduction. This paper is an attempt towards the development of the theory of differential equations of the form

$$d\lambda/d\mu = f(x, \lambda(\bar{S}_x))$$

where $(X, \mathscr{M}, \mu)$ is a measure space, $\bar{S}_x$ is a certain measurable set for each $x \in X$ and $d\lambda/d\mu$ denotes the Radon-Nikodym derivative of a complex measure λ (on the measurable space $(X, \mathscr{M})$) with respect to μ. Such equations include, as shown in §3, as special cases, ordinary differential equations and "measure differential equations" (as they are termed in **[1]**, **[4]**, **[5]**) of the form

$$Dy = f(x, y(x))Dg$$

where Dg is the distributional derivative of the right continuous real function g of bounded variation. In this paper existence and extension of solutions are treated.

For a measurable space $(X, \mathscr{M})$, $\mathrm{ca}(X, \mathscr{M})$ will denote, as in Dunford and Schwartz [**2**, p. 240], the space of all countably additive scalar (real or complex) functions (briefly, real measures or complex measures) on $\mathscr{M}$. (Note that real measures form a subclass of the complex ones, while positive measures do not do so since they include ∞ as an admissible value.) $\mathrm{ca}(X, \mathscr{M})$ is a Banach space where norm $\|\lambda\|$ is the total variation of λ on X (see Dunford and Schwartz [**2**, p. 161]). The total variation measure of a measure λ will be denoted by $|\lambda|$.

1. **Existence and uniqueness of solutions.** Let X be a linear space over the field $\mathscr{F}$ where $\mathscr{F}$ is the set $\boldsymbol{R}$ of real numbers or the set $\boldsymbol{C}$ of complex

Received by the editors July 12, 1971.

AMS 1970 *subject classifications*. Primary 34G05; Secondary 46G99.

Key words and phrases. Ordinary differential equations, measure differential equations, complex measure, Radon-Nikodym derivative, function of bounded variation, distributional derivative, total variation measure, principle of contraction mapping, Borel measure, Lebesgue-Stieltjes measure.

numbers. For each $x \in X$, define

$$S_x = \{\alpha x : -\infty < \alpha < 1\}, \qquad \bar{S}_x = \{\alpha x : -\infty < \alpha \leqq 1\} \quad \text{if } \mathscr{F} = \boldsymbol{R};$$

and

$$S_x = \{\alpha x : 0 < |\alpha| < 1\}, \qquad \bar{S}_x = \{\alpha x : 0 \leqq |\alpha| \leqq 1\} \quad \text{if } \mathscr{F} = \boldsymbol{C}.$$

Let $\mathscr{M}$ be a σ-algebra in X containing the sets $\bar{S}_x$ for all $x \in X$. Let μ be a positive σ-finite measure or a complex measure on $\mathscr{M}$. Let f be a scalar function defined on $S_\xi \times \Omega_a$ where $\xi \in X$ and

$$\Omega_a = \{\alpha : |\alpha| < a\}.$$

Assume that $f(x, \lambda(\bar{S}_x))$ is μ-integrable on S_ξ for each $\lambda \in \text{ca}(S_\xi, \mathscr{M}_\xi)$ where

$$\mathscr{M}_\xi = \{E \in \mathscr{M} : E \subset S_\xi\}.$$

Consider the equation

$$(*) \qquad d\lambda/d\mu = f(x, \lambda(\bar{S}_x))$$

where $d\lambda/d\mu$ denotes the Radon-Nikodym derivative of λ with respect to μ.

Definition 1. Let $\alpha_0 \in \Omega_a$, $x_0 \in S_\xi$, $\bar{S}_{x_0} \subset X_0 \in \mathscr{M}_\xi$ and let $\mathscr{M}_0$ be the smallest σ-algebra in X_0 containing $\bar{S}_{x_0} - S_{x_0}$ and the sets $\bar{S}_x$ for $x \in X_0 - S_{x_0}$ (obviously $\mathscr{M}_0 \subset \mathscr{M}_\xi$). A measure $\lambda \in \text{ca}(X_0, \mathscr{M}_0)$ will be called a solution of (*) on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$ if

(i) $\lambda(\bar{S}_{x_0}) = \alpha_0$,

(ii) $\lambda(E) \in \Omega_a$ for $E \in \mathscr{M}_0$,

(iii) $\lambda \ll \mu_0$ on $X_0 - S_{x_0}$ where μ_0 is the restriction of μ to $\mathscr{M}_0$ (i.e. $\mu_0(E) = 0$ implies $\lambda(E) = 0$ for $E \subset X_0 - S_{x_0}$, $E \in \mathscr{M}_0$),

(iv) λ satisfies (*) a.e. $[\mu_0]$ on $X_0 - S_{x_0}$.

The solution λ on X_0 with initial data $[\bar{S}_{x_0}, \alpha_0]$ will bc denoted, for the sake of convenience, by $\lambda[X_0; \bar{S}_{x_0}, \alpha_0]$. Clearly the conditions (iii) and (iv) in the above definition are equivalent to

$$\lambda(E) = \int_E f(x, \lambda(\bar{S}_x))\, d\mu_0 \quad \text{for } E \subset X_0 - S_{x_0}\ (E \in \mathscr{M}_0).$$

Theorem 1. *Let* $\alpha_0 \in \Omega_a$ *and* $x_0 \in S_\xi$. *There exists a unique solution* $\lambda_0 = \lambda_0[S_{x_1}; \bar{S}_{x_0}, \alpha_0]$ *of* (*) *for some* $x_1 \in S_\xi - \bar{S}_{x_0}$ *if the following conditions are satisfied*:

(i) $|\mu|(\bar{S}_{x_0} - S_{x_0}) = 0$;

(ii) *there exists a* μ-*integrable function* w *on* S_ξ *such that*

$$|f(x, \alpha)| \leqq w(x)$$

uniformly in $\alpha \in \Omega_a$;

(iii) f *satisfies a Lipschitz condition in* α; *i.e., given a set* $S_{x_1} \subset S_\xi$ *there* *exists a constant* $L=L(x_1)$ *such that*

$$|f(x, \alpha_1) - f(x, \alpha_2)| \leqq L\,|\alpha_1 - \alpha_2|$$

for all (x, α_1), $(x, \alpha_2) \in S_{x_1} \times \Omega_a$.

PROOF. Let r_n be a sequence of real numbers such that $r_n \downarrow 1$ and $S_{r_1 x_0} \supset S_{r_2 x_0} \supset S_{r_3 x_0} \supset \cdots \supset S_{x_0}$. Then

$$\bigcap_{n=1}^{\infty} (S_{r_n x_0} - \bar{S}_{x_0}) = \varnothing,$$

and therefore

$$|\mu|\,(S_{r_n x_0} - \bar{S}_{x_0}) \to 0.$$

We can therefore choose a real number r such that

$$\bar{S}_{x_0} \subset S_{rx_0}, \tag{1.1}$$

$$\int_{S_{rx_0} - \bar{S}_{x_0}} w(x)\, d\,|\mu| < a - |\alpha_0|, \tag{1.2}$$

and

$$L\,|\mu|\,(S_{rx_0} - \bar{S}_{x_0}) < 1, \tag{1.3}$$

where L is a Lipschitz constant for f on $S_{rx_0} \times \Omega_a$. It follows from condition (i) and (1.3) that

$$L\,|\mu|\,(S_{rx_0} - S_{x_0}) < 1. \tag{1.4}$$

Consider the space $\mathrm{ca}(S_{rx_0}, \mathscr{M}_0)$ where $\mathscr{M}_0$ is the smallest σ-algebra containing $\bar{S}_{x_0} - S_{x_0}$ and all the sets of the form $\bar{S}_x$ for $x \in S_{rx_0} - S_{x_0}$. Let Λ be the collection of all $\lambda \in \mathrm{ca}(S_{rx_0}, \mathscr{M}_0)$ with the properties:

$$\lambda(\bar{S}_{x_0}) = \alpha_0 \tag{1.5}$$

and

$$\|\lambda\| \leqq k \tag{1.6}$$

where

$$k = |\alpha_0| + \int_{S_{rx_0} - S_{x_0}} w(x)\, d\,|\mu| < a, \tag{1.7}$$

by (1.2) and condition (i). Clearly Λ is a closed, nonempty subset of $\mathrm{ca}(S_{rx_0}, \mathscr{M}_0)$ and is therefore a complete metric space. For each $\lambda \in \Lambda$, we have

$$|\lambda(E)| \leqq |\lambda|\,(E) \leqq \|\lambda\| \leqq k < a \quad \text{for } E \in \mathscr{M}_0. \tag{1.8}$$

Let T be the mapping defined on Λ by

$$(T\lambda)(E) = \alpha_0 \quad \text{for } E = \bar{S}_{x_0},$$
$$= \int_E f(x, \lambda(\bar{S}_x))\, d\mu \quad \text{for } E \subset S_{rx_0} - S_{x_0} \qquad (E \in \mathscr{M}_0).$$

Then $T\lambda \in \text{ca}(S_{rx_0}, \mathscr{M}_0)$, and

$$\|T\lambda\| = |\alpha_0| + \int_{S_{rx_0}-S_{x_0}} |f(x, \lambda(\bar{S}_x))|\, d\,|\mu|$$
$$\leqq |\alpha_0| + \int_{S_{rx_0}-S_{x_0}} w(x)\, d\,|\mu| \quad \text{by condition (ii)},$$
$$= k.$$

Therefore, $T\lambda \in \Lambda$. T thus maps Λ into itself. Furthermore, if $\lambda_1, \lambda_2 \in \Lambda$,

$$(T\lambda_1 - T\lambda_2)(E) = 0 \quad \text{for } E = \bar{S}_{x_0},$$
$$= \int_E [f(x, \lambda_1(\bar{S}_x)) - f(x, \lambda_2(\bar{S}_x))]\, d\mu$$
$$\text{for } E \subset S_{rx_0} - S_{x_0}\ (E \in \mathscr{M}_0).$$

Therefore,

$$\|T\lambda_1 - T\lambda_2\| = \int_{S_{rx_0}-S_{x_0}} |f(x, \lambda_1(\bar{S}_x)) - f(x, \lambda_2(\bar{S}_x))|\, d\,|\mu|$$
$$(1.9) \qquad \leqq L \int_{S_{rx_0}-S_{x_0}} |\lambda_1(S_x) - \lambda_2(S_x)|\, d\,|\mu|$$
$$\leqq L\,|\mu|\,(S_{rx_0} - S_{x_0})\, \|\lambda_1 - \lambda_2\|.$$

It follows from (1.4) and (1.9) that T is a contraction. Hence by the principle of contraction mapping, T has a unique fixed point λ_0. Also, $\lambda_0(E) \in \Omega_a$ by (1.8). λ_0 is then the solution of (*) on S_{rx_0} with initial data $[\bar{S}_{x_0}, \alpha_0]$. This completes the proof of Theorem 1.

2. **Extension of solution.** Let f be defined on $X \times \mathscr{F}$ and let the conditions of Theorem 1 be satisfied with S_ξ and Ω_a replaced by X and $\mathscr{F}$ respectively. Theorem 1 yields a solution $\lambda_0[X_0; \bar{S}_{x_0}, \alpha_0]$ where $\lambda_0 \in \text{ca}(X_0, \mathscr{M}_0)$, $X_0 \supset \bar{S}_{x_0}$ and $\lambda_0(\bar{S}_{x_0}) = \alpha_0$. Let $x_1 \in X_0 - \bar{S}_{x_0}$ be such that $|\mu|(\bar{S}_{x_1} - S_{x_1}) = 0$. There is a similar solution $\lambda_1[X_1; \bar{S}_{x_1}, \alpha_1]$ where $\lambda_1 \in \text{ca}(X_1, \mathscr{M}_1)$, $X_1 \supset X_0$ and $\lambda_1(\bar{S}_{x_1}) = \alpha_1$. Here $\mathscr{M}_0$ is the smallest σ-algebra containing $\bar{S}_{x_0} - S_{x_0}$ and sets of the form $\bar{S}_x$ for $x \in X_0 - S_{x_0}$ and $\mathscr{M}_1$ is the smallest σ-algebra containing $\bar{S}_{x_1} - S_{x_1}$ and sets of the form $\bar{S}_x$ for $x \in X_1 - S_{x_1}$. It follows from the uniqueness property that $\lambda_0(E) = \lambda_1(E)$ for $E \in \mathscr{M}_0 \cap \mathscr{M}_1$. Let $\mathscr{M}$ be the smallest σ-algebra containing the members of

$\mathscr{M}_0$ and $\mathscr{M}_1$. Let $\lambda \in \text{ca}(X_1, \mathscr{M})$ be such that

$$\begin{aligned} \lambda(E) &= \lambda_0(E) \quad \text{for } E \in \mathscr{M}_0, \\ &= \lambda_1(E) \quad \text{for } E \in \mathscr{M}_1. \end{aligned}$$

Then λ is a solution of (*) on the set X_1 such that $\lambda(\bar{S}_{x_0}) = \alpha_0$. We shall call λ the continuation of λ_0 to X_1. We thus extend the solution λ_0 to X_1. By repeating this process we arrive at a maximal set over which λ_0 is defined.

3. **Special cases.** (A) If $X = \boldsymbol{R}$, $\mathscr{F} = \boldsymbol{R}$ and μ=the Lebesgue measure m on $\boldsymbol{R}$, the equation (*) reduces to the equation

$$d\lambda/dm = f(x, \lambda((-\infty, x])) \tag{3.1}$$

which can be shown to be equivalent to the ordinary differential equation

$$dy/dx = f(x, y(x)). \tag{3.2}$$

More precisely, we shall prove the following:

THEOREM 2(A). *To each solution y of* (3.2) *with initial condition* $y(x_0) = \alpha_0$, *there corresponds a solution λ of* (3.1) *such that* $\lambda((-\infty, x_0]) = \alpha_0$, *and vice versa.*

PROOF. Let y_0 be a solution of (3.2) on $[x_0, x_1]$ with initial condition $y_0(x_0) = \alpha_0$. Define

$$\begin{aligned} y_1(x) &= 0 && \text{for } x \leqq x_0, \\ &= y_0(x) - \alpha_0 && \text{for } x_0 < x < x_1, \\ &= y_0(x_1) - \alpha_0 && \text{for } x \geqq x_1. \end{aligned}$$

Then $y_1 \in$NBV where NBV is the class of left-continuous functions φ of bounded variation such that $\varphi(x) \to 0$ as $x \to -\infty$, and hence there exists, by Rudin [**3**, Theorem 8.14(b)], a unique complex Borel measure λ_1 such that

$$y_1(x) = \lambda_1((-\infty, x)). \tag{3.3}$$

Since y_1 is absolutely continuous, $\lambda_1 \ll m$ (by Rudin [**3**, Theorem 8.16]). Let $\mathscr{M}_0$ be the smallest σ-algebra containing $\{x_0\}$ and the sets $(-\infty, x]$ for $x_0 \leqq x \leqq x_1$, and define λ_0 on $\mathscr{M}_0$ by

$$\lambda_0((-\infty, x_0]) = \alpha_0, \quad \lambda_0(E) = \lambda_1(E) \quad \text{for } E \subset [x_0, x_1] \ (E \in \mathscr{M}_0).$$

It is easy to see that $\lambda_0 \in \text{ca}([x_0, x_1], \mathscr{M}_0)$ and that $\lambda_0 \ll m_0$ where m_0 is the restriction of m to $\mathscr{M}_0$. Furthermore, for $x \in [x_0, x_1]$, we have

$$\begin{aligned} y_0(x) &= y_1(x) + \alpha_0 = \lambda_1((-\infty, x)) + \alpha_0 \\ &= \lambda_0((x_0, x)) + \lambda_0((-\infty, x_0]) = \lambda_0((-\infty, x)) \\ &= \lambda_0((-\infty, x]), \quad \text{since } \lambda_0 \ll m_0 \text{ and } m_0\{x\} = 0. \end{aligned} \tag{3.4}$$

Since y_0 is absolutely continuous, being a solution of (3.2) on $[x_0, x_1]$, y_0' $(\equiv dy_0/dx)$ is defined a.e. $[m]$ on $[x_0, x_1]$ and

$$y_0(x) = \alpha_0 + \int_{x_0}^{x} y_0'(t)\,dt \quad \text{for } x \in [x_0, x_1].$$

Therefore,

$$\lambda_0([x_0, x]) = \int_{[x_0,x]} y_0'(t)\,dt.$$

Thus,

$$y_0'(x) = d\lambda_0/dm_0 \quad \text{a.e. } [m]. \tag{3.5}$$

Now (3.4) and (3.5) show that λ_0 is a solution of (*) on $(-\infty, x_1]$ satisfying the initial condition $\lambda_0((-\infty, x_0]) = \alpha_0$.

Conversely, let λ_0 be a solution of (3.1) on $(-\infty, x_1]$ with initial condition $\lambda_0((-\infty, x_0]) = \alpha_0$. Let λ_1 be a complex Borel measure on $\boldsymbol{R}$ such that

$$\begin{aligned} \lambda_1((-\infty, x)) &= 0 && \text{for } x \leqq x_0, \\ &= \lambda_0((-\infty, x]) - \alpha_0 && \text{for } x_0 < x < x_1, \\ &= \lambda_0((-\infty, x_1]) && \text{for } x > x_1; \end{aligned}$$
$$\lambda_1(E) = \lambda_0(E) \quad \text{for measurable sets } E \subset [x_0, x_1].$$

Since $\lambda_0 \ll m_0$ on $[x_0, x_1]$, λ_0 being a solution of (3.1), it follows that $\lambda_1 \ll m$. Define y_1 by (3.3). By Rudin [**3**, Theorem 8.14(a) and Theorem 8.16], y_1 is absolutely continuous. Define

$$y_0(x) = y_1(x) + \alpha_0 \quad \text{for } x \in [x_0, x_1].$$

Then y_0 is absolutely continuous and

$$y_0(x) = \lambda_0((-\infty, x]) \quad \text{for } x \in [x_0, x_1].$$

Also, since

$$\lambda_0((x_0, x]) = \int_{(x_0,x]} \frac{d\lambda_0}{dm_0}(t)\,dt, \qquad x \in (x_0, x_1],$$

we have

$$y_0(x) = \alpha_0 + \int_{x_0}^{x} \frac{d\lambda_0}{dm_0}(t)\,dt.$$

Therefore,

$$\frac{d\lambda_0}{dm}(t) = y_0'(t) \quad \text{a.e. } [m] \text{ on } [x_0, x_1].$$

Thus, y_0 is a solution of (3.2) on $[x_0, x_1]$ satisfying $y_0(x_0) = \alpha_0$. This completes the proof of Theorem 2(A).

REMARK. In the special case considered above Theorem 1 reduces to a well-known local existence and uniqueness theorem for ordinary differential equations.

(B) Let $X=\boldsymbol{R}$, $\mathscr{F}=\boldsymbol{R}$ and $\mu=\mu_g$ where μ_g is the Lebesgue-Stieltjes measure induced by a right continuous function g of bounded variation. In this case the equation (*) takes the form

$$d\lambda/d\mu_g = f(x, \lambda((-\infty, x])). \tag{3.6}$$

Consider the equation

$$Dy = f(x, y(x))\, Dg \tag{3.7}$$

where Dg denotes the distributional derivative of g. The equation (3.7) is in fact equivalent (see [1], and also [4], [5]) to

$$y(x) = y(x_0) + \int_{(x_0,x]} f(s, y(s))\, dg. \tag{3.8}$$

By a solution y of (3.7) with initial condition $y(x_0)=\alpha_0$ is meant a right continuous function y of bounded variation such that y satisfies (3.8) and and $y(x_0)=\alpha_0$.

We shall prove the following:

THEOREM 2(B). *To each solution y of* (3.7) *with initial condition $y(x_0)=\alpha_0$, there corresponds a solution λ of* (3.6) *such that $\lambda((-\infty, x_0])=\alpha_0$, and vice versa.*

PROOF. Let y_0 be a solution of (3.7) on $[x_0, x_1]$ with initial condition $y_0(x_0)=\alpha_0$. Extend y_0 on $(-\infty, x_0)$ by defining $y_0(x)=0$ for $x\in(-\infty, x_0)$. Let $\mathscr{M}_0$ be the σ-algebra containing $\{x_0\}$ and the intervals $(-\infty, x]$ for $x\in[x_0, x_1]$. Let λ_{y_0} be the restriction to $\mathscr{M}_0$ of the Lebesgue-Stieltjes measure on $(-\infty, x_1]$ induced by y_0. Then

$$\begin{aligned} &\lambda_{y_0}((x', x'']) = y_0(x'') - y_0(x'), \qquad && x_0 \leqq x' < x'' \leqq x_1; \\ &\lambda_{y_0}((-\infty, x]) = y_0(x), && x \in [x_0, x_1]. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\lambda_{y_0}((x', x'']) = \int_{(x',x'']} f(x, \lambda_{y_0}((-\infty, x]))\, dg$$

and

$$\lambda_{y_0}((-\infty, x_0]) = y_0(x_0) = \alpha_0$$

which shows that λ_{y_0} is a solution of (3.6) with initial condition

$$\lambda_{y_0}((-\infty, x_0]) = \alpha_0.$$

Conversely, let λ_0 be a solution of (3.6) on $(-\infty, \beta]$ with initial condition $\lambda_{y_0}((-\infty, x_0])=\alpha_0$. Define $y_0(x)=\lambda_0((-\infty, x])$ for $x\in[x_0, \beta]$. Then

$$y_0(x) - y_0(x_0) = \int_{(x_0,x]} f(s, y_0(s))\, dg \quad \text{and} \quad y_0(x_0) = \alpha_0.$$

If $x_1>x_2>\cdots>x_n\to x$, then $y_0(x_n)\to y_0(x)$, since

$$(-\infty, x] = \bigcap_{n=1}^{\infty} (-\infty, x_n].$$

Thus y_0 is right continuous on $[x_0, \beta]$. If $x_0<x_1<\cdots<x_n=\beta$, then

$$\sum_{i=1}^{n} |y_0(x_i) - y_0(x_{i-1})| = \sum_{i=1}^{n} |\lambda_0((x_{i-1}, x_i])| \leqq |\lambda_0|\, ((-\infty, \beta))$$

so that

$$v(y_0, [x_0, \beta]) \leqq |\lambda_0|\, ((-\infty, \beta))$$

where $v(y_0, [x_0, \beta])$ denotes the total variation of y_0 on $[x_0, \beta]$. Since λ_0 is of bounded variation, the last inequality shows that y_0 is a function of bounded variation on $[x_0, \beta]$. Thus y_0 is a solution of (3.9) on $[x_0, x_1]$ with initial condition $y_0(x_0)=\alpha_0$. This completes the proof of Theorem 2(B).

REFERENCES

1. P. C. Das and R. R. Sharma, *On optimal controls for measure delay-differential equations*, SIAM J. Control **9** (1971), 43–61.

2. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York and London, 1958. MR **22** #8302.

3. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR **35** #1420.

4. W. W. Schmaedeke, *Optimal control theory for nonlinear vector differential equations containing measures*, SIAM J. Control **3** (1965), 231–280. MR **32** #7288.

5. R. R. Sharma and P. C. Das, *Existence and stability of measure differential equations*, Czechoslovak Math. J. (to appear).

DEPARTMENT OF MATHEMATICS, REGIONAL INSTITUTE OF TECHNOLOGY, JAMSHEDPUR, INDIA