THE BOOLEAN RING GENERATED BY A DISTRIBUTIVE LATTICE

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ABSTRACT. This note gives a simple characterization of the Boolean ring R generated by a distributive lattice L . The method is to introduce a Hausdorff topology on the Stone representation space E of L and note that R is the set of subsets of E which are compact and open with respect to this topology.

Let L be a distributive lattice. Since the adjunction of a least element to L , if there is not one already, is an order-theoretic triviality we assume L has a least element.

An *ideal* of L is a nonempty proper subset I of L such that $xy \in I$ if $x \in I$ and $y \in I$, and $y \in I$ if $x \in I$ and $y \leq x$. An ideal I of L is *prime* if $x \wedge y \notin I$ whenever $x \notin I$ and $y \notin I$. The *Stone space*, E , of L is the set of prime ideals of L equipped with the hull-kernel topology. A base for the topology of L is the set of all $U_a = \{\theta \in E : a \notin \theta\}$ ($a \in L$). The map $a \mapsto U_a$ is a lattice isomorphism of L onto the set of all compact open subsets of E .

Once we have L represented as a lattice of sets it is clear that L can be embedded in a Boolean ring. Indeed the representation by sets and subsequent embedding in a Boolean ring go back to Hausdorff [3], Birkhoff [2], Stone [6] and MacNeille [5]. Our variant of the process stems from recent work on structure spaces of lattice groups [1], a little of which has direct application here.

Consider the dual ordering of L . The set of prime dual ideals is exactly the set of complements of prime ideals of L . Thus E carries a second natural topology, the dual hull-kernel topology. A base for this is easily seen to be the set of all $V_a = \{\theta \in E : a \in \theta\} = E \sim U_a$ ($a \in L$).

Let E_S denote E with the supremum of the hull-kernel and dual hull-kernel topologies. It is trivial to check that E_S is a Hausdorff space. Our main result is:

THEOREM 1. *In E_S , U_a is compact ($a \in L$).*

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PROOF. By Alexander's subbase theorem [4] it is sufficient to show that if $\mathcal{C} = \{U_x : x \in X\} \cup \{V_y : y \in Y\}$ is a cover of U_a then a finite subset of $\mathcal{C}$ covers U_a . Suppose this is false.

We claim that the ideal I generated by X and the dual ideal J generated by $Y \cup \{a\}$ do not intersect. If they do there exist $x_1, \dots, x_n \in X$ and $y_1, \dots, y_m \in Y$ such that $x_1 \vee \dots \vee x_n \geq a \wedge y_1 \wedge \dots \wedge y_m$, then

$$U_{x_1} \cup \dots \cup U_{x_n} = U_{x_1 \vee \dots \vee x_n} \supset U_{a \wedge y_1 \wedge \dots \wedge y_m} = U_a \cap [U_{y_1} \cap \dots \cap U_{y_m}];$$

and hence $U_a \subset U_{x_1} \cup \dots \cup U_{x_n} \cup V_{y_1} \cup \dots \cup V_{y_m}$ contrary to hypothesis.

It follows that there is an ideal θ of L maximal with respect to $X \subset \theta$, $\theta \cap J = \emptyset$. Then if $d \notin \theta$, $b \notin \theta$, there exist $p, q \in \theta$ and $u, v \in J$ such that $d \vee p \geq u$ and $b \vee q \geq v$; then $(d \wedge b) \vee p \vee q \geq u \wedge v$ and $d \wedge b \notin \theta$. Hence θ is prime, $\theta \in U_a$ and θ is in no set of $\mathcal{C}$. Thus $\mathcal{C}$ is not a cover of U_a .

Let R denote the set of compact open subsets of E_S . If $M \in R$ then M is a union, and hence a finite union, of sets $U_a \cap V_b$ with $a, b \in L$. Thus R is the Boolean subring of $\mathcal{P}(E_S)$ generated by the set $\{U_a : a \in L\}$.

We can easily prove, using the techniques of [5, pp. 454, 455], the following result.

THEOREM 2. *If L is identified with its image in R then any lattice homomorphism of L into a Boolean ring K can be uniquely lifted to a Boolean ring homomorphism of R into K .*

This completes our description of the Boolean ring generated by L .

ADDED IN PROOF. These results were obtained independently but are essentially contained in a paper by H. A. Priestley, *Representation of distributive lattices by ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190. His approach and ours are different.

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