A CHARACTERIZATION OF GENERAL Z.P.I.-RINGS¹

KATHLEEN B. LEVITZ

ABSTRACT. A commutative ring A is a general Z.P.I.-ring if each ideal of A can be represented as a finite product of prime ideals. We prove that a commutative ring A is a general Z.P.I.-ring if each finitely generated ideal of A can be represented as a finite product of prime ideals. We also give a characterization of Krull domains in terms of $*$ -operations, as defined by Gilmer.

Introduction. A commutative ring A is a π -ring if each principal ideal of A can be represented as a product of prime ideals; A is a p -ring if each finitely generated ideal of A can be represented as a product of prime ideals. A commutative ring A is a *general Z.P.I.-ring* if each ideal of A can be represented as a product of prime ideals. Mori characterized π -rings in [4], [5] and general Z.P.I.-rings in [6]. In [9] Wood extended Mori's results on π -rings. Wood also gave a characterization of general Z.P.I.-rings which is independent of Mori's work [10].

Let D be an integral domain with identity and with quotient field K . If $F(D)$ is the set of nonzero fractional ideals of A , a mapping $B \rightarrow B^*$ of $F(D)$ into $F(D)$ is called a *$*$ -operation on D* if the following three conditions hold for any a in $K - \{0\}$, and any B, C in $F(D)$.

- (1) $(a)^* = (a)$, $(aB)^* = aB^*$.
- (2) $B \subseteq B^*$, if $B \subseteq C$, $B^* \subseteq C^*$.
- (3) $(B^*)^* = B^*$.

If $(B)^* = (C)^*$, B and C are called *$*$ -equivalent*, denoted $B \sim_* C$. If $B = B^*$, B is called a *$*$ -ideal*. An integral domain D is a Krull domain if $D = \bigcap V_\alpha$, where $\{V_\alpha\}$ is a set of rank one discrete valuation rings with the property that for each nonzero x in D , $xV_\alpha = V_\alpha$ for all but a finite number of the V_α .

In the first section of this paper, we give a new characterization of Krull domains. We prove that an integral domain A with identity is a Krull domain if and only if there is a $*$ -operation on A such that each nonzero principal ideal of A is $*$ -equivalent to a product of prime ideals. Then we consider π -rings with no zero divisors, called π -domains. Theorem 1.2 states that A is a π -domain with identity if and only if A is a Krull domain

Received by the editors June 28, 1971.

AMS 1970 subject classifications. Primary 13F05, 13F10.

Key words and phrases. General Z.P.I.-ring, Dedekind domain, Krull domain, $*$ -operation, special primary ring.

¹ Research supported by the National Science Foundation grant GP 19406.

© American Mathematical Society 1972

in which each minimal prime ideal is invertible. We also prove that if A is a π -domain with identity, then each generalized quotient ring of A is also a π -domain with identity.

In §2, we consider p -rings, as well as p -domains, those p -rings with no zero divisors. Clearly, any Dedekind domain is a p -domain with identity. In Theorem 2.1, we show that any p -domain with identity is a Dedekind domain. Using Theorem 2.1, we prove the main result of this section, Theorem 2.3. This theorem states that a necessary and sufficient condition for a ring A to be a general Z.P.I.-ring is that A be a p -ring.

1. π -rings. Let A be an integral domain with identity. An example of a $*$ -operation on A is the v -operation which associates with each nonzero fractional ideal B of A the intersection of all the principal fractional ideals of A containing B . This intersection is denoted $\text{div}(B)$; $\text{div}(B)$ is called a *divisorial ideal*.

THEOREM 1.1. *Let A be an integral domain with identity. A is a Krull domain if and only if there exists a $*$ -operation on A such that each nonzero principal ideal of A is $*$ -equivalent to a product of prime ideals.*

PROOF. ($\rightarrow$) Let A be a Krull domain. By [7, Theorem 1], $v: B \rightarrow \text{div}(B)$ is a $*$ -operation on A such that each nonzero principal ideal of A is v -equivalent to a product of prime ideals of A .

($\leftarrow$) Let $a \in A - \{0\}$. $(a) \sim_* C$, where C is a finite product of prime ideals of A . By [3, Theorem 28.1], $C \subseteq (a) \subseteq \text{div}(C)$. It follows that $(a) \sim_v \text{div}(C)$. Since (a) is a divisorial ideal, $(a) = \text{div}(C)$. By [8, Theorem 3.1], A is a Krull domain.

If D is an integral domain, we call a prime ideal P of D a *minimal prime* of D if P is of height 1.

THEOREM 1.2. *Let A be an integral domain with identity. A is a π -domain if and only if A is a Krull domain in which each minimal prime ideal is invertible.*

PROOF. ($\rightarrow$) Since each nonzero principal ideal of A can be represented as a product of prime ideals, each nonzero principal ideal of A is v -equivalent to a product of prime ideals. This implies that A is a Krull domain by Theorem 1.1. That each minimal prime ideal of A is invertible follows from [10, Theorem 1.1.3].

($\leftarrow$) Let d be a nonzero nonunit of A . (d) can be represented as a finite intersection of primary ideals belonging to minimal primes of A because A is a Krull domain [3, Corollary 35.10]. Let $(d) = \bigcap_{i=1}^n Q_i$, where for each i , Q_i is P_i -primary and P_i is a minimal prime ideal of A . Since each P_i is invertible, $Q_i = P_i^{e_i}$ for some nonnegative integer e_i [2, Theorem 6.6].

Hence $(d) = \bigcap_{i=1}^n P_i^{e_i}$, where for each i , P_i is a minimal prime of A . We can assume that all the P_i are distinct. Then by [3, Lemma 35.15],

$$(d) = \bigcap_{i=1}^n P_i^{e_i} = \prod_{i=1}^n P_i^{e_i}.$$

Hence A is a π -domain.

Let A be an integral domain with quotient field K . If A' is a ring such that $A \subset A' \subset K$, A' is an overring of A . Let A be an integral domain with identity and with quotient field K . Let B be an overring of A . B is called a *generalized quotient ring of A* if B is flat as an A -module.

THEOREM 1.3. *Let A be a π -domain with identity. Each generalized quotient ring of A is a π -domain.*

PROOF. Let B be a generalized quotient ring of A . Each prime ideal of A extends in B either to B or to a proper prime ideal of B [1, Theorem 2]. If $b \in B - \{0\}$, $b = a_1/a_2$, where a_1, a_2 are in $A - \{0\}$. Since a_1A and a_2A can be represented as finite products of prime ideals of A , bA can also be represented as a finite product of prime ideals of A ; $bA = \prod_{i=1}^n P_i^{e_i}$ where for each i , P_i is a prime ideal of A and $e_i \in \mathbb{Z} - \{0\}$. It follows that

$$bB = \left(\prod_{i=1}^n P_i^{e_i} \right) B = \prod_{i=1}^n (P_i B)^{e_i},$$

which is a representation of bB as a product of prime ideals of B .

2. p -rings. We recall that a *Dedekind domain* is an integral domain with identity such that each ideal is representable as a product of prime ideals. A principal ideal ring A with identity is called a *special primary ring* if A contains only one prime ideal $M \neq A$ and if $M^k = (0)$ for some positive integer k .

THEOREM 2.1. *Let A be an integral domain with identity and with quotient field K . A is a Dedekind domain if and only if A is a p -domain.*

PROOF. ($\rightarrow$) By definition of Dedekind domain.

($\leftarrow$) Every p -ring is a π -ring. By Theorem 1.2, A is a Krull domain in which each minimal prime ideal is invertible. To conclude that A is Dedekind it suffices for us to prove that A is of Krull dimension 1. Each prime ideal of a Krull domain contains a minimal prime ideal. It follows that if each minimal prime ideal is maximal, A is Dedekind.

Let P be any minimal prime ideal of A . Let $a \in A - P$. Since (P, a) is finitely generated, $(P, a) = \prod_{i=1}^n P_i$ and $(P, a^2) = \prod_{j=1}^m Q_j$, where for each i and j , P_i and Q_j are prime ideals of A . Let $\bar{A} = A/P$, and let $\bar{a}$ be the residue class of a modulo P . Then $(\bar{a}) = \prod_{i=1}^n (P_i/P)$, and $(\bar{a}^2) = \prod_{j=1}^m (Q_j/P)$. Since $(\bar{a})$ is an invertible ideal of $\bar{A}$, $\{P_i/P\}_{i=1}^n$ and $\{Q_j/P\}_{j=1}^m$ are invertible prime

ideals of $\bar{A}$. By [11, Lemma 5, p. 272], $m=2n$, and the Q_j 's can be renumbered in such a way that

$$Q_{2i}/P = Q_{2i-1}/P = P_i/P.$$

Thus $Q_{2i}=Q_{2i-1}=P_i$, and $(P, a^2)=(P, a)^2$. Now $P \subset (P, a)^2 \subset (P^2, a)$. For any $x \in P$, $x=y+za$ where $y \in P^2$ and $z \in A$. Since $za \in P$, $z \in P$. This implies that $P \subset P^2 + Pa$. The reverse inclusion, $P^2 + Pa \subset P$, is clear. It follows that

$$P = P^2 + Pa = P(P + (a)).$$

Recalling that P is invertible, we conclude that $A=P+(a)$, and that P is a maximal ideal. Therefore, each minimal prime ideal is a maximal ideal.

LEMMA 2.2. *Let A be a p -ring. If B is any finitely generated ideal of A , then A/B is a p -ring.*

PROOF. Let $\bar{C}$ be a finitely generated proper ideal of A/B . $\bar{C}$ is the image of a finitely generated ideal C which contains B . $C = \prod_{i=1}^n P_i^{e_i}$, where for each i , P_i is a prime ideal of A containing B . Then $\bar{C} = \prod_{i=1}^n (P_i/B)^{e_i}$.

Using Theorem 2.1 and Lemma 2.2 we now prove the main result of this paper.

THEOREM 2.3. *A is a general Z.P.I.-ring if and only if A is a p -ring.*

PROOF. ($\rightarrow$) By definition of general Z.P.I.-ring.

($\leftarrow$) We consider three cases: (a) A is a commutative ring with identity, (b) A is a commutative ring without identity, but with zero divisors; and (c) A is an integral domain without identity.

(a) A is a finite direct sum of π -domains with identity and special primary rings by [4, Hauptsatz]. Using [9, Theorem 2], we can conclude A is a general Z.P.I.-ring if any summand A_j of A that is a domain is Dedekind. Let $A = \sum_{i=1}^n A_i$ be a representation of A as a direct sum of π -domains with identity and special primary rings. If e_i is the identity of the summand A_i , $A_i = Ae_i$. Thus for any j , A_j is isomorphic to $A/(\{e_i\}_{i \neq j})$. By Lemma 2.2, A_j is a p -ring. By Theorem 2.1, if A_j is a domain, A_j is Dedekind.

(b) By [5, Hauptsatz II], $A=M$ or $A=M+K$, where K is a field and M is a ring without identity such that each ideal of M is a power of M . A is a general Z.P.I.-ring by [9, Theorem 2].

(c) It is clear from the definition of a p -ring that a Noetherian p -ring is a general Z.P.I.-ring. We now show that if the p -ring A is an integral domain without identity, then A is Noetherian. To prove that A is Noetherian we show that for any nonzero element $d \in A$, $A/(d^2)$ is Noetherian. If $a \in A - \{0\}$, $A/(d^2)$ is a commutative ring with zero divisors. By Lemma 2.2, $A/(d^2)$ is a p -ring. Since $A/(d^2)$ is contained in either the class of p -rings considered in (a) or the class considered in (b), $A/(d^2)$ is a general Z.P.I.-ring.

All general Z.P.I.-rings are Noetherian by [10, Theorem 2.1.21]. Hence $A/(d^2)$ is Noetherian.

REFERENCES

1. T. Akiba, *Remarks on generalized rings of quotients*, Proc. Japan Acad. **40** (1964), 801–806. MR **31** #4807.
2. R. Gilmer, *Multiplicative ideal theory*. Chap. I: *Basic concepts*, Queen's Papers in Pure and Appl. Math., no. 12, Queen's Univ. Press, Kingston, Ontario, 1968. MR **37** #5198.
3. ———, *Multiplicative ideal theory*. Chap. II: *Integral extension*, Queen's Papers in Pure and Appl. Math., no. 12, Queen's Univ. Press, Kingston, Ontario, 1968. MR **37** #5198.
4. S. Mori, *Über die Produktzerlegung der Hauptideale*. III, J. Sci. Hiroshima Univ. Ser. A **10** (1940), 85–94. MR **1**, 327.
5. ———, *Über die Produktzerlegung der Hauptideale*. IV, J. Sci. Hiroshima Univ. Ser. A **11** (1941), 7–14. MR **2**, 343.
6. ———, *Allgemeine Z.P.I.-Ringe*, J. Sci. Hiroshima Univ. Ser. A **10** (1940), 117–136. MR **2**, 121.
7. T. Nishimura, *Unique factorization of ideals in the sense of quasi-equality*, J. Math. Kyoto Univ. **3** (1963), 115–125. MR **28** #109.
8. S. Tramel, *Factorization of principal ideals in the sense of quasi-equality*, Doctoral Dissertation, Louisiana State University, Baton Rouge, La., 1968.
9. C. Wood, *On general Z.P.I.-rings*, Pacific J. Math. **30** (1969), 837–846. MR **40** #1380.
10. ———, *On general Z.P.I.-rings*, Doctoral Dissertation, Florida State University, Tallahassee, Fla., 1967.
11. O. Zariski and P. Samuel, *Commutative algebra*. Vol. 1, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1958. MR **19**, 833.

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306

Current address: Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506