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CLOSED IDEALS IN SUBALGEBRAS OF BANACH ALGEBRAS. I

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ABSTRACT. In this note we obtain an elementary extension of the so-called Fundamental Theorem of Segal Algebras to a special class of noncommutative Banach algebras. In particular the case of noncommutative group algebras is settled.

In this note the ideal structure of a class of Banach algebras that subsume Cigler's normed ideals [**1**]—and consequently Reiter's Segal algebras [**4**, p. 127]—is examined.

In §1 the axioms of our algebras (abstract Segal algebras) are stated. We then construct a generalization of the Fundamental Theorem of Segal Algebras for commutative group algebras—[**4**, pp. 129–130]. Our result also takes care of the noncommutative case. We also show that approximate identities need not be bounded. Specializing to the commutative case, in §2 we compute the space of regular maximal ideals for our algebras—this result contains the corresponding result in [**1**, p. 277]. For some concrete examples see [**2**], [**5**], [**6**]. Applying the results of §1, in §3 we briefly discuss the relationship between Beurling algebras (see [**4**, p. 83] for definitions) and our algebras. Specifically, Beurling algebras are not normed ideals. Moreover, the condition of "Beurling-Domar" is not necessary for the results in [**1**, p. 280] and [**4**, 3.6 (ii), pp. 137–138]. We conclude with reference to the work of Martin and Yap [**3**] and Yap [**6**] for "counterexamples".

1. **Fundamental Theorem of Abstract Segal Algebras.** In this section no commutativity hypothesis is assumed and "ideal" means "right ideal"—of course the same results hold for left ideals.

DEFINITION 1.1. $(B, \| \|')$ is an abstract Segal algebra (ASA) with respect to (wrt) a Banach algebra $(A, \| \|)$ if and only if

(1) B is a dense ideal in A and B is a Banach algebra wrt the norm $\| \|'$.

(2) $\exists M>0$ so that $\|f\| \leqq M \|f\|'$, $\forall f \in B$.

(3) $\exists C>0$ so that $\|fg\|' \leqq C \|f\| \, \|g\|'$, $\forall f, g \in B$.

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When there is no possibility for confusion we shall say "B is an ASA" without further qualifications.

REMARK. By an application of (2) and (3), (1) can be replaced with "B is dense in A and B is a Banach space wrt $\| \ \|'$" and recapture (1).

DEFINITION 1.2. A Banach algebra $(X, \| \ \|)$ has right approximate units if and only if, for every $\varepsilon>0$ and for each $f\in X$, $\exists e(f)\in X$ for which $\|fe(f)-f\|<\varepsilon$.

We shall say that a Banach algebra has Property $P(r, 1)$ if every right approximate unit is also a left approximate unit—this does not mean that $fe(f)=e(f)f$! Property $(Pr, 1)$ is needed to establish the results below for left ideals. Clearly if a Banach algebra has a right approximate identity, then it also has right approximate units. Note that we do not require an approximate unit to be bounded in any sense!

Here is the main theorem of this note.

THEOREM 1.1. *If B is an ASA satisfying Property $P(r, 1)$, then the following two statements are true.*

(a) *If J is a closed ideal in A, then $J\cap B$ is a closed ideal in B.*

(b) *If I is a closed ideal in B, then* $\mathrm{cl}(I)$ (= *closure of I in A*) *is a closed ideal in A and* $I=\mathrm{cl}(I)\cap B$.

PROOF. For (a) to hold it is sufficient that B be a subalgebra of A that is a Banach space wrt $\| \ \|'$ and (2) of Definition 1.1 hold. The proof is carried out by observing that $J\cap B$ is complete wrt $\| \ \|+\| \ \|'$ and this latter norm is equivalent to $\| \ \|'$. For the first part of (b) choose $f\in\mathrm{cl}(I)$, $g\in A$ and sequences $\langle f_n\rangle$ in I, $\langle g_n\rangle$ in B so that $\|f_n-f\|\to 0$ and $\|g_n-g\|\to 0$. That $fg\in\mathrm{cl}(I)$ follows from the joint continuity of multiplication in A and I is an ideal in B. Clearly $I\subset\mathrm{cl}(I)\cap B$, thus to complete the proof it suffices to establish the reverse inclusion. Choose $f\in\mathrm{cl}(I)\cap B$ and let $\varepsilon>0$ be given. Choose $e(f)\in B$ so that $\|fe(f)-f\|'<\varepsilon/2$. There exists a sequence $\langle f_n\rangle$ in I so that $\|f_n-f\|\to 0$ and, by (3) of Definition 1.1,

$$\|f_ne(f) - fe(f)\|' \leqq c \, \|f_n - f\| \, \|e(f)\|'$$

and hence $\|f_ne(f)-fe(f)\|'\to 0$. Thus we can choose a positive integer i so that $\|f_ie(f)-fe(f)\|'<\varepsilon/2$. A routine application of the triangle inequality obtains $\|f_ie(f)-f\|'<\varepsilon$ so that $f\in I$ and the proof is complete.

The next theorem shows that approximate identities need not be bounded.

THEOREM 1.2. *Suppose B is an ASA with a right approximate identity $\langle e(\alpha)\rangle$. If B is a proper subset of A, then $\langle e(\alpha)\rangle$ is not bounded in the B norm.*

PROOF. Suppose $\|e(\alpha)\|' \leqq K$ for all α. Then by (2) and (3) of Definition 1.1

$$(1/C)\|f\| \leqq \|f\|' \leqq MK\,\|f\|, \qquad \forall f \in B.$$

But this latter estimate means $\|\ \|$ and $\|\ \|'$ are equivalent on B which is a contradiction since B is a *proper* dense subset of A.

We now prepare a lemma which will be used to characterize the maximal ideal space of a commutative ASA.

LEMMA 1.1. *Suppose A is a normed algebra and let B be a subalgebra of A that is also a normed algebra wrt the norm $\|\ \|'$. If $\exists C > 0$ so that*

$$\|fg\|' \leqq C\,\|f\|\,\|g\|', \qquad \forall f, g \in B,$$

then

$$\lim(\|f^n\|')^{1/n} \leqq \|f\|, \qquad \forall f \in B.$$

PROOF. If $f=0$ the result is trivial, so suppose $0 \neq f \in B$. For each n, $\|f^n\|' = \|f^{n-1}f\|'$ and consequently $\|f^n\|' \leqq C\|f\|^{n-1}\|f\|'$. Taking nth roots, and passing to the limit wrt n obtains the desired result.

NOTATION. For any normed algebra X, write $S(f, X)$ for $\lim\|f^n\|^{1/n}$.

2. **Regular maximal ideal space of a commutative ASA.** If B is a commutative ASA with an approximate identity then, via Theorem 1.1, there is a 1-1 correspondence between the closed ideal of B and those of A. However, no information is provided concerning the regular maximal ideals of B. Lemma 1.1 is the key for this latter information.

THEOREM 2.1. *If B is an ASA wrt a commutative Banach algebra A, then the regular maximal ideal spaces of B and A are homeomorphic. Moreover, B is semisimple if and only if A is semisimple.*

PROOF. If h is a complex homomorphism on B, then $|h(f)| \leqq S(f, B)$, $\forall f \in B$, and an application of Lemma 1.1 obtains $|h(f)| \leqq \|f\|$, $\forall f \in B$. Thus h is a complex homomorphism on B wrt to the restricted A norm. Since B is dense in A, h has a unique extension to a complex homomorphism defined on all of A. Since $\|f\| \leqq M\|f\|'$, the restriction to B of bounded functionals on A are also bounded wrt $\|\ \|'$. Since regular maximal ideals are uniquely determined by complex homomorphisms it follows that the regular maximal ideal spaces of B and A are isomorphic as sets. A routine argument shows that the "two" Gelfand topologies coincide to complete the proof. The semisimplicity part is a direct consequence of the above construction of complex homomorphisms.

3. **Some remarks on Beurling algebras.** Throughout this section we work with a fixed group algebra, $L^1(G, \lambda)$, where G is a locally compact Hausdorff group with a right Haar measure λ.

THEOREM 3.1. *If $B_1(G) \subset B(G)$ are Beurling algebras, then $B_1(G)$ is an ASA wrt $B(G)$ if and only if $B_1(G) = B(G)$.*

PROOF. If $B_1 = B$ the result is clear. Suppose B_1 is a proper dense subset of B. Now, any right approximate identity for $L^1(G, \lambda)$ is also a right approximate identity for *any* Beurling algebra on G—[**4**, p. 84]. Straightforward calculations show that these approximate identities are bounded in any Beurling algebra norm. An application of Theorem 1.2 shows that (3) of Definition 1.1 fails and the result follows.

COROLLARY. *The above result is true with $B(G) = L^1(G, \lambda)$.*

REMARKS. If $B(G)$ is a Beurling algebra, then Theorem 1.1 carries over to ASA's wrt $B(G)$ without change. This is a considerable improvement over the corresponding result in [**1**, p. 280] since there the condition of Beurling-Domar [**4**, p. 132] is imposed on $B(G)$ along with the commutativity of G. One can also apply (a) of Theorem 1.1 to a pair of Beurling algebras $B_1 \subset B$. This latter remark eliminates the Beurling-Domar condition required in [**4**, 3.6 (ii), pp. 137–138].

As a final remark we note that, even in view of Theorems 1.1 and 2.1, ASA's wrt group algebras can fail to inherit important properties of group algebras. For more on these matters see [**3**], [**6**].

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