ON THE EXISTENCE OF INVARIANT SUBSPACES IN SPACES WITH INDEFINITE METRIC¹

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ABSTRACT. Let P_1, P_2 be complementary projections in Hilbert space H . Let U be a one-to-one and onto operator in H with $Q(Ux) = Q(x)$, where $Q(x) = \|P_1x\|^2 - \|P_2x\|^2$. The sufficient condition is given for the unique existence of maximal subspace L invariant under all operators commuting with U , and such that $Q(x) \geq 0, x \in L$. The result was obtained in the course of attacking the problem proposed by Phillips [5] (see also [1]).

Let H be a (separable) complex Hilbert space, with the ordinary scalar product $(\cdot, \cdot)$, and with indefinite metric $Q(x, y) = (Jx, y)$ ($J = P_1 - P_2$; P_1, P_2 are two complementary orthogonal projections). Let $\mathcal{K}$ be the set of all bounded linear operators K from H to P_2H ($\equiv H_2$) with a bound ≤ 1 and such that $KH_2 = 0$. Let $\mathcal{M}$ be the set of all maximal positive subspaces M in H ; M is said to be positive if $Q(x, x) \geq 0$ for x in M . Let $\mathcal{C}(T)$ be the set of all bounded linear operators that commute with T and its Q -adjoint operator T° ; T° is defined by $Q(T^\circ x, y) = Q(x, Ty)$. U is called a Q -unitary operator if $Q(Ux, Uy) = Q(x, y)$ and if U is onto and one-to-one. Let $H_j = P_jH$ ($j = 1, 2$). Let $U_{jk} = P_jUP_k$ ($j, k = 1, 2$). Now our main result is the following:

THEOREM. *Let U be a Q -unitary operator. Suppose that there exist convex open sets Ω_1, Ω_2 with $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$ ($\bar{\Omega}$ means the closure of Ω) such that*

$$(*) \quad \begin{aligned} &\{(U(1 + K)x_1, x_1); \|x_1\| = 1, x_1 \in H_1\} \subset \Omega_1, \\ &\{((1 - K)Ux_2, x_2); \|x_2\| = 1, x_2 \in H_2\} \subset \Omega_2, \end{aligned}$$

for all K in $\mathcal{K}$. Then there exists a unique maximal positive subspace invariant under T in $\mathcal{C}(U)$.

COROLLARY. *Suppose that A is a bounded Q -selfadjoint operator ($A = A^\circ$) satisfying the condition (*), given in the above theorem, with U replaced by A . Then there exists a unique maximal positive subspace invariant under T in $\mathcal{C}(A)$.*

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REMARK. The following example shows that there exist, generally, more than one invariant subspace in $\mathcal{M}$ when the condition (*) is violated. Let H be a two-dimensional complex euclidean space with element $x = \{x_1, x_2\}$. Define projections $P_1x = \{x_1, 0\}$, $P_2x = \{0, x_2\}$, and let A be an operator defined by $Ax = \{\lambda x_1 + x_2, -x_1\}$ (λ :real). Then the A_λ satisfies all the conditions in the corollary if $|\lambda| > 2$. Hence for $|\lambda| > 2$ there exists a unique maximal positive subspace M_λ invariant under $\mathcal{C}(A_\lambda)$. On the other hand, by direct calculation, we can see that the set $\mathcal{M}_\lambda$ of all maximal positive subspaces invariant under $\mathcal{C}(A_\lambda)$ consists of $\{[x, (-\lambda + (\lambda^2 - 4)^{1/2})x/2]; x \in \mathbb{C}\}$ for $\lambda > 2$; $\{[x, (-\lambda \pm (\lambda^2 - 4)^{1/2})x/2]; x \in \mathbb{C}\}$ for $|\lambda| \leq 2$; $\{[x, (-\lambda - (\lambda^2 - 4)^{1/2})x/2]; x \in \mathbb{C}\}$ for $\lambda < -2$.

PROOF OF THEOREM. By the Q -unitarity of U ,

$$\begin{aligned} \|x\|^2 + \|(U_{21} + zU_{22}K)x\|^2 &\leq |z|^2 \|Kx\|^2 + \|(U_{11} + zU_{12}K)x\|^2 \\ &\leq |z|^2 \|x\|^2 + \|(U_{11} + zU_{12}K)x\|^2. \end{aligned}$$

Hence,

$$(1) \quad (1 - |z|^2) \|x\|^2 + \|(U_{21} + zU_{22}K)x\|^2 \leq \|(U_{11} + zU_{12}K)x\|^2$$

for x in H_1 , K in $\mathcal{K}$ and $|z| \leq 1$. Using (1), we shall show that for each K in $\mathcal{K}$ and $|z| < 1$, $(U_{11} + zU_{12}K)^{-1}$ exists as a bounded operator on H_1 with a bound

$$(2) \quad \|(U_{11} + zU_{12}K)^{-1}\| \leq (1 - |z|^2)^{-1/2}.$$

Indeed, $U_{11} + zU_{12}K$ is, by (1), one-to-one.

If y is orthogonal to $R(U_{11} + zU_{12}K)$ ($R(T)$ = the range of T), then

$$\begin{aligned} 0 &= (y, (U_{11} + zU_{12}K)x) = Q(y, U(1 + zK)x) \\ &= Q(U^\circ y, (1 + zK)x) = ([P_1U^\circ - \bar{z}K^*P_2U^\circ]y, x) \end{aligned}$$

for all x in H_1 ; K^* is the adjoint operator of K with respect to the ordinary scalar product (note that $K^*H_1 = 0$, $R(K^*) \subseteq H_1$, and $\|K^*\| \leq 1$). Hence $[P_1U^\circ - \bar{z}K^*P_2U^\circ]y = 0$, from which it follows that $\|P_1U^\circ y\| \leq |z| \|K^*\| \cdot \|P_2U^\circ y\| \leq \|P_2U^\circ y\|$. Hence $\|y\|^2 = Q(y, y) = Q(U^\circ y, U^\circ y) = \|P_1U^\circ y\|^2 - \|P_2U^\circ y\|^2 \leq 0$. Here we used the Q -unitarity of U° implied by that of U . Hence $y = 0$, i.e., the range of $U_{11} + zU_{12}K$ is dense in H_1 . Thus $U_{11} + zU_{12}K$ is, by (1), onto. The desired bound of $(U_{11} + zU_{12}K)$ follows from (1).

By Phillips [5], $M \in \mathcal{M}$ if and only if $M = \{x_1 + Kx_1; x_1 \in H_1\}$ for some K in $\mathcal{K}$. Then it is easy to see that M is invariant under U if and only if K satisfies

$$(3) \quad KU_{12}K + KU_{11} - U_{22}K - U_{21} = 0.$$

To find a solution K of (3) in $\mathcal{K}$, we consider instead the equation:

$$(4) \quad zK(z)U_{12}K(z) + K(z)U_{11} - zU_{22}K(z) - U_{21} = 0$$

in $|z| < 1$ and construct $K(z)$ from a sequence of holomorphic functions $\{K_j(z)\}$. We set

$$K_0(z) = 0 \quad \text{and} \quad K_{j+1}(z) = (U_{21} + zU_{22}K_j(z)) \cdot (U_{11} + zU_{12}K_j(z))^{-1}$$

i.e.,

$$(5) \quad K_{j+1}(z)(U_{11} + zU_{12}K_j(z)) = U_{21} + zU_{22}K_j(z).$$

Then the $K_j(z)$ has the properties:

- (i) $K_j(z) \in \mathcal{K}$;
- (ii) $K_j(z)$ is holomorphic in $|z| < 1$;
- (iii) for any compact set Δ in $|z| < 1$ $\sup_{\Delta} \|K'_j(z)\|$ is uniformly bounded in j (K' : derivative in z of K);
- (iv) $K_j(z)$ converges uniformly to some bounded operator $\tilde{K}(z)$ on some small disk $|z| < \varepsilon$ in the operator norm.

Indeed, if $K_j(z) \in \mathcal{K}$, then we have, by (1),

$$\|K_{j+1}(U_{11} + zU_{12}K_j)x\|^2 = \|(U_{21} + zU_{22}K_j)x\|^2 \leq \|(U_{11} + zU_{12}K_j)x\|^2,$$

showing (i). Suppose that $K_j(z)$ is holomorphic in $|z| < 1$. Then from (2) it is easy to see that $(U_{11} + zU_{12}K_j)^{-1}$ is holomorphic in $|z| < 1$. Hence $(U_{21} + zU_{22}K_j)(U_{11} + zU_{12}K_j)^{-1}$ is holomorphic in z , showing (ii). (iii) is a direct consequence of (ii). Since

$$\begin{aligned} K_{j+1}(z) - K_j(z) &= zU_{22}(K_j - K_{j-1})(U_{11} + zU_{12}K_j)^{-1} \\ &\quad + z^2(U_{21} + zU_{22}K_{j-1})(U_{11} + zU_{12}K_{j-1})^{-1} \\ &\quad \cdot U_{12}(K_j - K_{j-1})(U_{11} + zU_{12}K_j)^{-1}, \end{aligned}$$

we have, by (2) and (i),

$$\|K_{j+1} - K_j\| \leq \frac{1}{2} \|K_j - K_{j-1}\|$$

for sufficiently small $|z| \leq \varepsilon$. Hence $K(z)$ converges to some $\tilde{K}(z) \in \mathcal{K}$ uniformly on $|z| \leq \varepsilon$ in the operator norm. Thus we have (iv).

By (i), (ii), (iii), there exists a subsequence $K_{j'}(z)$ converging (in the weak topology) uniformly on any compact set in $|z| < 1$ to some holomorphic (in $|z| < 1$) operator-valued function $K(z) \in \mathcal{K}$. By (iv), $\tilde{K}(z) = K(z)$ in $|z| < \varepsilon$. Since $K_j(z)$ converges strongly to $\tilde{K}(z)$ in $|z| < \varepsilon$ and since K_j satisfies (5), $\tilde{K}(z)$, and so $K(z)$, satisfies (4) in $|z| < \varepsilon$. Since $K(z)$ is weakly, and so strongly, holomorphic in $|z| < 1$, $K(z)$ must satisfy (4) in the whole unit disk $|z| < 1$. Now we claim:

(v) $K'(z)$ is uniformly bounded for real z in $0 < z < 1$.

(vi) Equation (3) has at most one solution.

For the moment we assume that (v) and (vi) hold. Then $K(z)$ has a strong limit $K(1) \in \mathcal{K}$ for $z \rightarrow 1$ ($0 < z < 1$). Letting $z \rightarrow 1$ in (4), we have

$$K(1)U_{12}K(1) + K(1)U_{11} - U_{22}K(1) - U_{21} = 0.$$

Thus the subspace $M = \{x_1 + K(1)x_1; x_1 \in H_1\} \in \mathcal{M}$ is invariant under U . Besides M is invariant also under T in $\mathcal{C}(U)$. Indeed, if T is in $\mathcal{C}(U)$, then $T + T^\circ$ is a Q -selfadjoint operator in $\mathcal{C}(U)$; note that $T^\circ \in \mathcal{C}(U)$ if $T \in \mathcal{C}(U)$. Hence the operator $V(t)$ defined by $V(t) = \exp[it(T + T^\circ)]$ is Q -unitary. Hence $V(t)$ maps M onto M . Since $UV(t)M = V(t)UM \subset V(t)M$, the space $V(t)M$ in $\mathcal{M}$ is invariant under U . By the uniqueness of the solution of (6) (see vi), which implies the uniqueness of invariant maximal positive subspace, we have $V(t)M = M$. Hence $(it)^{-1}[V(t) - I]M \subset M$. Letting $t \rightarrow 0$, we have $(T + T^\circ)M \subset M$. Similarly $i(T - T^\circ)M \subset M$. Thus $TM \subset M$.

PROOF OF (v). Since $\bar{\Omega}_1, \bar{\Omega}_2$ are closed, convex and disjoint sets, there exist θ ($-\pi/2 \leq \theta \leq \pi/2$), α_1, α_2 ($\alpha_1 > \alpha_2$) such that

$$(6) \quad \operatorname{Re}(e^{i\theta}\zeta_1) > \alpha_1 \quad \text{and} \quad \operatorname{Re}(e^{i\theta}\zeta_2) < \alpha_2,$$

or

$$(7) \quad \operatorname{Re}(e^{i\theta}\zeta_1) < \alpha_2 \quad \text{and} \quad \operatorname{Re}(e^{i\theta}\zeta_2) > \alpha_1$$

for ζ_1 in $\bar{\Omega}_1$ and ζ_2 in $\bar{\Omega}_2$. We consider only the case (6); the case (7) can be treated similarly. Then by the assumption (*),

$$\operatorname{Re}(e^{i\theta}P_1U(1 + K)x_1 - \alpha_1x_1, x_1) \geq 0,$$

$$\operatorname{Re}(e^{i\theta}P_2(1 - K)Ux_2 - \alpha_2x_2, x_2) \leq 0,$$

for all $K \in \mathcal{K}$, $x_1 \in H_1$ and $x_2 \in H_2$. By this "dissipativity condition",

$$(8) \quad \begin{aligned} \|\exp[-te^{i\theta}P_1U(1 + zK) + t\alpha_1]P_1\| &\leq 1; \\ \|\exp[te^{i\theta}P_2(1 - K)U - t\alpha_2]P_2\| &\leq 1 \end{aligned}$$

for all $t > 0$, $K \in \mathcal{K}$ and $0 < z \leq 1$. Differentiation in z of (4) gives

$$K'(z)(U_{11} + zU_{12}K(z)) + z(K(z)U_{12} - U_{22})K'(z) = B(z)$$

i.e.,

$$K'(z)P_1U(1 + zK(z))P_1 + zP_2(K(z) - 1)UP_2K'(z) = B(z),$$

where $B(z) = -K(z)U_{12}K(z) + U_{22}K(z)$. Hence

$$K'(z) = -e^{-i\theta} \lim_{N \rightarrow \infty} \int_0^N e^{-(\alpha_1 - z\alpha_2)t} \exp[te^{i\theta}P_2(1 - K(z))UP_2 - \alpha_2zt]$$

$$\cdot P_2 \cdot B(z) \cdot \exp[-te^{i\theta} \cdot P_1U(1 + zK(z))P_1 + \alpha_1t]P_1 dt.$$

Hence, by (8),

$$\|K'(z)\| \leq \|B(z)\| \cdot (\alpha_1 - \alpha_2)^{-1} \leq (\alpha_1 - \alpha_2)^{-1}(\|U_{12}\| + \|U_{22}\|)$$

for $0 < z < 1$.

PROOF OF (vi). We consider only the case (6). Let K_1, K_2 be solutions of (3) in $\mathcal{K}$. Then $K_0 = K_1 - K_2$ satisfies $K_0(U_{12}K_2 + U_{11}) = (U_{22} - K_1U_{12})K_0$.

Hence we have

$$\begin{aligned}
 K_0 &= -e^{-i\theta} \int_0^N \frac{d}{dt} \{ \exp[te^{i\theta}P_2(1 - K_1)UP_2]P_2K_0 \\
 &\quad \cdot \exp[-te^{i\theta}P_1U(1 + K_2)]P_1 \} dt \\
 &\quad + e^{-i\theta} e^{-(\alpha_1 - \alpha_2)N} \exp[Ne^{i\theta}P_2(1 - K_1)UP_2 - \alpha_2 N] \\
 &\quad \cdot P_2K_0 \exp[-Ne^{i\theta}P_1U(1 + K_2)P_1 + \alpha_1 N]P_1 \\
 &= e^{-i\theta - (\alpha_1 - \alpha_2)N} \exp(\cdots) K_0 \exp(\cdots).
 \end{aligned}$$

Letting $N \rightarrow \infty$, we have, by (9), $K_0 = 0$. Thus the proof is completed.

PROOF OF COROLLARY. The Q -selfadjoint operator A defines the one-parameter family of group $U(t) = e^{itA}$ of Q -unitary operators. Then we have

$$U(t) = I + (itA) + \frac{1}{2!} (itA)^2 + \cdots.$$

Hence there exists a t_0 such that, for each $0 < t < t_0$, $U(t)$ satisfies the condition (*). By the theorem, for each $0 < t < t_0$ there exists the unique subspace M in $\mathcal{M}$ invariant under $\mathcal{C}(U(t))$. Since $\mathcal{C}(A) \subset \mathcal{C}(U(t))$, M is invariant under $\mathcal{C}(A)$. Conversely if $M (\in \mathcal{M})$ is invariant under $\mathcal{C}(A)$, then it is also invariant under $\mathcal{C}(U(t))$ by the above expansion. Hence the M is the unique subspace in $\mathcal{M}$ invariant under $\mathcal{C}(A)$.

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