A VECTOR MEASURE WITH NO DERIVATIVE

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ABSTRACT. Given a nonatomic scalar measure μ , there is a vector valued, μ -continuous measure of finite variation which has no derivative with respect to μ , but which has the property that the closure of its range is compact and convex.

In this note we give an example of a vector measure which answers a question raised in [5]. Further, the example can be used to gain some information about certain Banach spaces with the Radon-Nikodym property. Measure theoretic terminology is that of [5]. The Banach space (under supremum norm) of all real sequences which converge to zero is written c_0 . An element of c_0 will be denoted by a doubly indexed sequence $(a_{n,i})$, with $n \geq 1$ and $2^n \leq i < 2^{n+1}$.

THEOREM. Suppose that (S, Σ, μ) is a finite nonnegative measure space which has no atoms. There is a measure $\varphi: \Sigma \rightarrow c_0$ such that

- (1) φ is μ -continuous and of finite variation,
- (2) the closure of the range of φ is compact and convex, and
- (3) φ has no Bochner integrable derivative with respect to μ .

PROOF. Since μ has no atoms we may generate a doubly indexed sequence $(A_{n,i})$, $n \geq 1$ and $2^n \leq i < 2^{n+1}$, of measurable sets such that $\mu(A_{n,i}) = 2^{-n}\mu(S)$ and $A_{n,i}$ is the disjoint union of $A_{n+1,2i}$ and $A_{n+1,2i+1}$. Define $\varphi: \Sigma \rightarrow c_0$ by $\varphi(A) = (\mu(A \cap A_{n,i}))$. Clearly $\|\varphi(A)\| \leq \mu(A)$ for all measurable A , so φ must be μ -continuous and of finite variation.

The range of φ is contained in the set M of sequences $(a_{n,i})$ such that $0 \leq a_{n,i} \leq 2^{-n}\mu(S)$ and $a_{n,i} = a_{n+1,2i} + a_{n+1,2i+1}$ for $2^n \leq i < 2^{n+1}$. The sequences in M converge to zero uniformly so M is relatively compact, and it is clear that M is closed and convex. For $(a_{n,i}) \in M$ and $\varepsilon > 0$, choose $m \geq 1$ so that $2^{-m}\mu(S) < \varepsilon$. Again using the fact that μ has no atoms there are measurable sets $B_i \subset A_{m,i}$, $2^m \leq i < 2^{m+1}$, satisfying $\mu(B_i) = a_{m,i}$. Then $\|\varphi(B) - (a_{n,i})\| < \varepsilon$, where $B = \bigcup_{2^m \leq i < 2^{m+1}} B_i$.

To see that φ has no derivative with respect to μ , we suppose the contrary. Let $f: S \rightarrow c_0$ be a derivative of φ and write $(e_{n,i})$ for the unit vectors

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in l_1 . For each $i \geq 2$ and $A \in \Sigma$,

$$\int_A \langle f(s), e_{n,i} \rangle \mu(ds) = \mu(A \cap A_{n,i}),$$

so there is, for each i , a μ -null set $C_i \subset S$ such that $\langle f(s), e_{n,i} \rangle = \chi_{A_{n,i}}(s)$ for $s \notin C_i$. Choose $t \in S \setminus \bigcup_i C_i$. By the way in which the sets $(A_{n,i})$ were chosen $\chi_{A_{n,i}}(t) = 1$ for infinitely many indices i , so $\lim_i \langle f(t), e_{n,i} \rangle \neq 0$. This is the desired contradiction.

A Banach space E is said to have the Radon-Nikodym property (rn) if, given any finite measure space (S, Σ, μ) , every μ -continuous measure of finite variation $\varphi: \Sigma \rightarrow E$ has a Bochner integrable derivative with respect to μ (the classical examples are the reflexive spaces, separable duals and $l_1(\Gamma)$). Little is known about the structure of such Banach spaces. The example given above indicates that no Banach space E with a subspace isomorphic to c_0 has this property. Stated affirmatively [1] this means that each weakly unconditionally Cauchy sequence in E is unconditionally convergent. It is known [4] that a complemented subspace of $L_1(\mu)$ which has (rn) is isomorphic to $l_1(\Gamma)$ for some set Γ . The example given above shows that a quotient of $C(S)$ (S compact Hausdorff) with (rn) must be reflexive [3], and that a complemented subspace of $C(S)$ with (rn) must be finite dimensional. Finally, let us note that this example, combined with the Radon-Nikodym theorem of Dunford and Pettis [2] for measures into separable duals, gives an easy proof of the classical result of Orlicz that no separable dual contains c_0 .

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