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ON CHARACTERIZATION OF RIEMANNIAN MANIFOLDS BY GROWTH OF TUBULAR NEIGHBORHOODS[1]

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ABSTRACT. If the area function of the tubular neighborhoods of a compact submanifold of a Riemannian manifold satisfies a certain linear differential inequality, then the codimension of the submanifold is at most the order of that inequality.

1. Let N be a Riemannian manifold of dimension $n \geqq 2$ and let M be a compact orientable submanifold of dimension m embedded in N. (All manifolds, maps, etc. are supposed smooth.) For $s>0$, let M_s denote the set of points lying on geodesics normal to M and at arc length s from M. For sufficiently small s, M_s is a smooth hypersurface in N. We denote by $\mathscr{A}(s)$ the area of M_s. H. Wu **[2]** derived an elegant formula for $\mathscr{A}''(s)$ and used it to characterize the Euclidean plane amongst Riemannian manifolds. Later he and R. A. Holzsager **[1]** proved the following more encompassing characterization:

A Riemannian manifold has the property that the growth function $\mathscr{A}$ of each one of its compact hypersurfaces satisfies the linear differential equation

$$\mathscr{A}''(s) + c\mathscr{A}(s) = 0$$

(*where c is a fixed constant*) *if and only if it is a two-dimensional Riemannian manifold of constant curvature equal to c.*

In this note we obtain a formula for $\mathscr{A}''(s)$ and indeed for all the derivatives of $\mathscr{A}$, valid for submanifolds M of any dimension. Our method yields $\mathscr{A}''$ in a simpler, more mechanical fashion than does Wu's. We can easily reprove the theorem of Holzsager and Wu. We also obtain an extension of their theorem to the case when $\mathscr{A}$ satisfies a linear differential *inequality* of higher order with "nice" coefficients. We show the order of this differential inequality is an upper bound for the codimensions of the submanifolds involved.

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2. Let M be a fixed compact orientable submanifold of N with an embedded tubular neighborhood U in N. Let $U^*=U-N$. Our arguments are local, so we may suppose N to be oriented. Choose a positively oriented (local) orthonormal frame field $e_1, \cdots, e_n$ on U^*, supposed *adapted* so e_1 is the tangent vector of geodesics leaving M normally, while each e_i $(2 \leqq i \leqq n)$ is parallel along such a geodesic. Let $\omega_1, \cdots, \omega_n$ be the dual coframe and let ω_{ij} $(1 \leqq i, j \leqq n)$ be the connection 1-forms of the Riemannian structure of N as restricted to U^*. The forms ω_i, ω_{ij} satisfy the Cartan structural equations

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j$$

and

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},$$

where Ω_{ij} is the curvature 2-form.

For sufficiently small $s>0$, $M_s \subset U^*$ and is a hypersurface with normal field e_1 so there are functions $h_{ij}=h_{ji}$ $(2 \leqq i, j \leqq n)$ on U^* with

$$\omega_{1i} = \sum_{j=2}^{n} h_{ij}\omega_j.$$

The *mean curvature* is given at $p \in M_s$ by

$$H(p) = \frac{1}{n-1} \sum_{j=2}^{n} h_{jj}(p).$$

(Note that the signs have been chosen so that M_1 has $H=1$ when M is the origin in the Euclidean plane.) Finally, the function $\mathscr{A}(s)$ is given by

$$\mathscr{A}(s) = \int_{M_s} \omega_2 \wedge \cdots \wedge \omega_n.$$

We use Stokes' theorem on the closed set $M_{s+\delta,s}$ included between M_s and $M_{s+\delta}$ $(\delta>0)$. We then have

$$\mathscr{A}(s+\delta) - \mathscr{A}(s) = \int_{M_{s+\delta,s}} d(\omega_2 \wedge \cdots \wedge \omega_n).$$

Using the first Cartan structural equation, we calculate

$$d(\omega_2 \wedge \cdots \wedge \omega_n) = (n-1)H\omega_1 \wedge \cdots \wedge \omega_n.$$

Hence

$$\mathscr{A}(s+\delta) - \mathscr{A}(s) = \int_{M_{s+\delta,s}} (n-1)H\omega_1 \wedge \cdots \wedge \omega_n.$$

Keeping in mind the geometrical meaning of ω_1, we divide by δ and let $\delta \to 0$ to obtain

$$\mathscr{A}'(s) = \int_{M_s} (n-1)H\omega_2 \wedge \cdots \wedge \omega_n.$$

In order to calculate higher derivatives of $\mathscr{A}$, it will be convenient to use this notation: if F is a smooth function on U^*, then

$$dF = \sum_{j=1}^{n} F_{,j}\omega_j.$$

Again applying Stokes' theorem, we have

$$\mathscr{A}'(s+\delta) - \mathscr{A}'(s) = \int_{M_{s+\delta,s}} (n-1)\, d(H\omega_2 \wedge \cdots \wedge \omega_n).$$

It follows easily that

$$\mathscr{A}''(s) = \int_{M_s} (n-1)\{H_1 + (n-1)H^2\}\omega_2 \wedge \cdots \wedge \omega_n.$$

There is a general formula:

$$\mathscr{A}^{(k)}(s) = \int_{M_s} I_k\omega_2 \wedge \cdots \wedge \omega_n.$$

The functions I_k satisfy the recurrence relation

$$I_{k+1} = I_{k,1} + (n-1)HI_k.$$

We will not attempt to give an explicit formula for I_k in terms of H and its derivatives.

3. Now we will extend the theorem of Holzsager and Wu. The following estimate generalizes the lemma of Wu [**2**].

LEMMA 1. *As $s \to 0$, $H \sim (n-m-1)/(n-1) \cdot 1/s$. This asymptotic relation is uniform from one normal geodesic to another and may be differentiated as often as desired.*

(Here we use the asymptotic symbol $\sim$ in the usual sense: $f \sim g$ means $f(s)/g(s) \to 1$ as $s \to 0$.)

Verification of the asymptotic relation of Lemma 1 along a given normal geodesic proceeds from an elementary calculation using the first and second Cartan structural equations. We obtain relations expressing the successive derivatives of *each* h_{ii} in terms of the lower-order radial derivatives of h_{ii} as well as the Riemannian curvature and its derivatives. Using the smoothness of the data, we establish inductively that each radial derivative of h_{ii} has the proper order of infinity as $s \to 0$. Knowing this, we can use the uniqueness of asymptotic expansion to calculate the asymptotic behavior of the derivatives of H as $s \to 0$. The compactness of M allows the conclusion that these asymptotic estimates are uniform over M from one normal geodesic to another. The perturbation techniques used are by now standard in global analysis.

Using the recurrence relation for I_k and Lemma 1, we can prove an estimate for I_k by induction.

LEMMA 2. *As* $s \to 0$,

$$I_k \sim (n-m-1)\cdots(n-m-k)s^{-k},$$

and this estimate is uniform from one normal geodesic to another.

Here is our main result.

THEOREM. *Let k be a positive integer and fix $m \leqq n-2$. Suppose for each compact M of dimension m there are bounded continuous functions $c_1^M, \cdots, c_k^M$ such that the area function $\mathscr{A}$ of M satisfies the differential inequality*

$$\mathscr{A}^{(k)}(s) + \sum_{j=1}^{k} c_j^M(s)\mathscr{A}^{(k-j)}(s) \leqq 0$$

for sufficiently small $s>0$. Then $n-m \leqq k$.

The proof is based upon Lemma 2. We have

$$\mathscr{A}^{(j)}(s) = \int_{M_s} I_j \omega_2 \wedge \cdots \wedge \omega_n$$

whence the differential inequality hypothesized implies

$$\int_{M_s} \left\{ I_k + \sum_{j=1}^{k} c_j^M(s) I_{k-j} \right\} \omega_2 \wedge \cdots \wedge \omega_n \leqq 0.$$

By Lemma 2, the function in curly brackets has the uniform asymptotic expression

$$\{\cdots\} = (n-m-1)\cdots(n-m-k)s^{-k} + o(s^{-k})$$

as $s \to 0$. If $n-m>k$, $\{\cdots\}>0$ for sufficiently small $s>0$, so

$$\int_{M_s} \{\cdots\} \omega_2 \wedge \cdots \wedge \omega_n > 0.$$

This contradiction shows $n-m \leqq k$, as was claimed.

4. We show how to obtain the theorem of Holzsager and Wu. Suppose

$$\mathscr{A}''(s) + c\mathscr{A}(s) = 0$$

for some constant c. We apply our theorem with submanifolds M consisting of single points, so $m=0$. We conclude $n=2$. Calculating I_2 explicitly, we find

$$\mathscr{A}''(s) + c\mathscr{A}(s) = \int_{M_s} (H_1 + H^2 + c)\omega_2.$$

In our case, M_s is a curve. In terms of the adapted coframe field, $\omega_{12}= H\omega_2$. The Gauss curvature function K satisfies

$$d\omega_{12} = -K\omega_1 \wedge \omega_2$$

from which we obtain $H_1+H^2=-K$. Therefore,

$$\int_{M_s} (c - K)\omega_2 = 0.$$

Since M and s are arbitrary, $K\equiv c$ on N.

Bibliography

1. R. A. Holzsager and H. Wu, *A characterization of two-dimensional Riemannian manifolds of constant curvature*, Michigan Math. J. **17** (1970), 297–299.

2. H. Wu, *A characteristic property of the euclidean plane*, Michigan Math. J. **16** (1969), 141–148. MR **39** #7544.

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