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BI-UNITARY PERFECT NUMBERS

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ABSTRACT. Let d be a divisor of a positive integer n. Then d is a *unitary divisor* if d and n/d are relatively prime, and d is a *bi-unitary divisor* if the greatest common unitary divisor of d and n/d is 1. An integer is *bi-unitary perfect* if it equals the sum of its proper bi-unitary divisors. The purpose of this paper is to show that there are only three bi-unitary perfect numbers, namely 6, 60 and 90.

A divisor d of an integer n is a *unitary divisor* if d and n/d are relatively prime. A divisor d of an integer n is a *bi-unitary divisor* if the greatest common unitary divisor of d and n/d is 1. Let $\sigma(n)$ be the sum of the divisors of n, let $\sigma^*(n)$ be the sum of the unitary divisors of n, and let $\sigma^{**}(n)$ be the sum of the bi-unitary divisors of n.

We say that N is *unitary perfect* if $\sigma^*(N)=2N$. Subbarao and Warren [2] showed that 6, 60, 90 and 87360 are the first four unitary perfect numbers; Wall reported [3] that

$$146{,}361{,}946{,}186{,}458{,}562{,}560{,}000 = 2^{18}3\cdot 5^4 7\cdot 11\cdot 13\cdot 19\cdot 37\cdot 79\cdot 109\cdot 157\cdot 313$$

is also unitary perfect and later showed [4] it to be the next such number after 87360. Subbarao [1] has conjectured that there are only finitely many unitary perfect numbers.

We say that N is *bi-unitary perfect* if $\sigma^{**}(N)=2N$. The purpose of this paper is to show that the first three unitary perfect numbers, i.e., 6, 60 and 90, are the *only* bi-unitary perfect numbers.

One easily verifies that σ^{**} is multiplicative and that if p is prime and $e\geqq 1$, then

$$\sigma^{**}(p^e) = \sigma(p^e) = (p^{e+1} - 1)/(p - 1)$$

if e is odd, and

$$\sigma^{**}(p^e) = (p^{e+1} - 1)/(p - 1) - p^{e/2}$$

if e is even. Hence $\sigma^{**}(n)\leqq\sigma(n)$ with equality if and only if every prime which divides n does so an odd number of times. It also follows immediately that $\sigma^{**}(n)$ is odd if and only if n is 1 or a power of 2; consequently, each odd prime power unitary divisor of n contributes at least one factor 2 to $\sigma^{**}(n)$.

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Let ϕ be Euler's totient.

THEOREM 1. *There are no odd bi-unitary perfect numbers.*

PROOF. If $\sigma^{**}(N)=2N$ and N is odd, then N must be a prime power, say $N=p^e$. But then

$$\sigma^{**}(p^e)/p^e \leqq \sigma(p^e)/p^e < p^e/\phi(p^e) = p/(p-1) \leqq \tfrac{3}{2},$$

so N cannot be bi-unitary perfect.

We define $f(n)=\sigma^{**}(n)/n$ and note that if p is any prime, then

$$1 = f(1) < f(p^2) < f(p) < f(p^4) < f(p^3),$$

and

$$f(p^\alpha) < f(p^{\alpha+2}) < p/(p-1)$$

for all natural numbers α, and $f(p^\alpha)>f(p^{\alpha+1})$ if α is odd. Also, we remark that $f(N)=2$ if and only if N is bi-unitary perfect.

THEOREM 2. *The only even bi-unitary perfect numbers are* 6, 60 *and* 90.

PROOF. Henceforth we assume that N is bi-unitary perfect and even, and write $N=2^aM=2^aXY$ where $X=\sigma^{**}(2^a)$. Our scheme of proof is to establish that: (i) if a is odd, then $a=1$; (ii) if $a=1$, then $N=6$ or $N=90$; (iii) if $a=2$, then $N=60$; (iv) if $a\geqq 6$, then $(N, 3)=1$; (v) $a=4$ and $a=6$ are impossible; and (vi) $a\geqq 8$ is impossible.

If a is odd, then $3|X$. If $a\geqq 3$, then $f(2^a)\geqq \frac{15}{8}$, so

$$f(N) \geqq f(8)f(9) = (\tfrac{15}{8})(\tfrac{10}{9}) = \tfrac{25}{12} > 2,$$

contradicting the assumption that $f(N)=2$. Thus (i) is proved.

If $a=1$, then $3|N$ and N has at most two distinct odd prime divisors. If $3^3|N$, then

$$f(N) \geqq (\tfrac{3}{2})(\tfrac{112}{81}) > 2.$$

If $3\|N$, then $N = 6$. If $3^2\|N$, then $5|N$ as $1+3^2=2\cdot 5$ and 5 cannot divide N exactly twice without N having three distinct odd prime divisors. Then $f(N)>2$ unless $5\|N$, which yields $N=90$. Thus (ii) is proved.

If $a=2$, then $5|N$ and N has at most three distinct odd prime divisors. Suppose $3\nmid N$, and let $N=4M$ with M odd. Then since

$$f(N) < 5M/4\phi(M)$$

and

$$(5\cdot 7\cdot 11\cdot 13)/(4\cdot 6\cdot 10\cdot 12) < (5\cdot 5\cdot 11\cdot 13)/(4\cdot 4\cdot 10\cdot 12)$$
$$< (5\cdot 5\cdot 7\cdot 13)/(4\cdot 4\cdot 6\cdot 12) < 2,$$

we must have $N=2^2 5^b 7^c 11^d$ for some choice of positive exponents b, c and d. Moreover, $A=\sigma^{**}(11^d)$ must be divisible by 2 exactly once and not by 3, which requires that d be even, say $d=2e$. Then

$$A = (1 + 11^{e+1})(11^e - 1)/10$$

and 5 and 7 are the only odd primes which can divide A. From congruences modulo 7, it is clear that if $7|A$ then $3|e$; but then the factor (11^e-1) is a multiple of 19, so $19|A$, a contradiction. Now, A cannot be a power of 2, because to avoid having $1+11^{e+1}$ be a multiple of 3, we must have e odd, whence $(11^e-1)/10$ is odd. If $7\nmid A$, then $5|A$, so $5^2|(11^e-1)$ and consequently $5|e$; but then $3221|A$, a contradiction. Thus $3|N$.

If $a=2$ and $3|N$ we write $N=2^2 3^b 5^c N'$ where $(N', 30)=1$; in fact, N' is either 1 or a prime power. If neither b nor c is 2, then

$$f(N) \geqq (\tfrac{5}{4})(\tfrac{4}{3})(\tfrac{8}{5}) = 2$$

with equality only for $N=60$. If $c=2$ then $13|N'$, and 13 cannot divide N exactly twice or else $17|N$ and N has at least four odd prime divisors. If $b\geqq 5$ then

$$f(N) \geqq (\tfrac{5}{4})(\tfrac{1066}{729})(\tfrac{26}{25})(\tfrac{14}{13}) > 2;$$

if b is 1 or 3 there are too many factors 2 in the numerator of $f(N)$, and if $b=4$, there is an excess factor 7; if $b=2$, then

$$f(N) \leqq (\tfrac{5}{4})(\tfrac{10}{9})(\tfrac{26}{25})(\tfrac{13}{12}) < 2.$$

Thus we cannot have $c=2$. If $b=2$, then as $5^2|\sigma^{**}(2^2 3^2)$ we must have $c\geqq 2$. So $c\geqq 3$ as the case $b=c=2$ has already been eliminated. If $7\nmid N$ then

$$f(N) < (\tfrac{5}{4})(\tfrac{10}{9})(\tfrac{5}{4})(\tfrac{11}{10}) < 2.$$

Thus, $7|N$. If $c=4$, then N has too many factors 3. The possibilities $7\|N$, $7^3\|N$ and $7^4\|N$ lead to too many factors 2 in N, while $7^2\|N$ implies that $f(N)<2$. Therefore, $7^5|N$ and if $c=3$ or $c\geqq 5$, then

$$f(N) \geqq (\tfrac{5}{4})(\tfrac{10}{9})f(5^6 7^6) > 2.$$

Hence (iii) is proved.

If N is divisible by $3\cdot 2^6$ then

$$f(N) \geqq (\tfrac{119}{64})(\tfrac{10}{9}) > 2,$$

contradicting the fact that $f(N)=2$. Thus (iv) is proved.

If $a=4$, then $N=16\cdot 27N'$ with N' odd. Then

$$f(N) \geqq (\tfrac{27}{16})(\tfrac{112}{81}) > 2,$$

a contradiction. If $a=6$, then $7\cdot 17|M$. If 7 does not divide M exactly twice, then $f(N)>2$; thus $7^2\|M$, so $5^2|M$. If $5^3|M$ then $f(N)>2$. Hence

$5^2 \| M$, so $13 | M$ and 13 does not divide M exactly twice or else $5^3 | M$. Then

$$f(N) \geqq \left(\tfrac{119}{64}\right)\left(\tfrac{50}{49}\right)\left(\tfrac{26}{25}\right)\left(\tfrac{14}{13}\right)\left(\tfrac{290}{289}\right) > 2,$$

a contradiction which establishes (v).

If $a \geqq 8$, then $(N, 5)=1$ or else $f(N) \geqq \left(\tfrac{495}{256}\right)\left(\tfrac{26}{25}\right) > 2$, a contradiction. We set $a=2b$ with $b \geqq 4$. Then

$$X = \sigma^{**}(2^a) = 2^{2b+1} - 2^b - 1 = (2^b - 1)(1 + 2^{b+1})$$

is composite. Since $(3, M)=1$ and $X | M$, and $1+2^{b+1}=2(2^b-1)+3$, we know the factors 2^b-1 and $1+2^{b+1}$ are relatively prime.

Let p be any prime dividing X, and suppose $p^c \| M$. If $c \neq 2$, then

$$1 + p^{-1} \leqq f(p^c) \leqq f(M) = 2^{2b+1}/(2^{2b+1} - 2^b - 1),$$

which requires that $p \geqq 2^{b+1}-3+2/(1+2^b) > X^{1/2}$, whence $p=X$, contradicting the fact that X is composite. Thus any prime that divides X must divide M exactly twice. Then

$$1 + p^{-2} = f(p^2) \leqq f(M)$$

requires that $p^4 > X$. Hence X has no more than three distinct prime factors, and $(X, 30)=1$. But $X=(2^b-1)(1+2^{b+1})$, so one of the factors must be prime. As $(X, 3)=1$, we must have b odd. With the restriction that b be odd, either

$$2^b - 1 \equiv 1 \pmod{10} \quad \text{and} \quad 1 + 2^{b+1} \equiv 5 \pmod{10}$$

or

$$2^b - 1 \equiv 1 + 2^{b+1} \equiv 7 \pmod{10}.$$

The first case is eliminated as $(N, 5)=1$. In the second case, one of the two numbers must be prime: call this prime p. Then $p \equiv 7 \pmod{10}$ and $p^2 \| N$. But $\sigma^{**}(p^2)=1+p^2$ is then a multiple of 5, so $5 | N$, a contradiction.

Hence the theorem is proved.

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