BAND DECOMPOSITIONS OF SEMIGROUPS

MOHAN S. PUTCHA AND JULIAN WEISSGLASS

ABSTRACT. The existence of a band decomposition of a semi-group into semigroups having at most one idempotent and a group ideal whenever it has an idempotent is investigated. It is shown that S has such a band decomposition if and only if, for every $c, d \in S^1$, $a \in S$, the idempotent multiples of cad and ca^2d coincide. The main result is used to characterize those semigroups which are bands of groups, extending a theorem of Clifford.

1. **Introduction.** The primary purpose of this paper is to investigate band decompositions of semigroups into semigroups having at most one idempotent and a group ideal whenever it has an idempotent. One result of this investigation is a characterization of those semigroups which are bands of groups, extending a theorem of Clifford (cf. [1] or [2, §4.2, Exercise 3]).

Throughout this paper S will denote a semigroup and $E = E(S)$ the set of idempotents of S. The set-valued function $\mathfrak{W}$ on S is defined by $\mathfrak{W}(a, E(S)) = Sa \cap aS \cap E$. $\mathfrak{W}(a, E(S))$ can be thought of as the set of idempotent multiples of a in S. We shall write $\mathfrak{W}(a)$ for $\mathfrak{W}(a, E(S))$ when there is no possibility of confusion. Any terms which are not defined in this article can be found in Clifford and Preston [2].

The concept of a viable semigroup was introduced in [3]. S is *viable* if $ab, ba \in E$ implies $ab = ba$. We proved in [3] that S is viable if and only if S is a semilattice of $\mathscr{S}$-indecomposable semigroups each of which contains at most one idempotent and a group ideal whenever it has an idempotent.

A semigroup S is *weakly viable* if for every $c, d \in S^1$, $a \in S$, $\mathfrak{W}(cad) = \mathfrak{W}(ca^2d)$. That a viable semigroup is weakly viable follows from the corollary to Theorem 6.

The main theorem (Theorem 6) of this paper proves, in part, that S is weakly viable if and only if it is a band of semigroups having at most one idempotent and a group ideal whenever it has an idempotent. As a consequence of this result, the structure of certain weakly viable semigroups is determined. For example, Theorem 7 proves that S is weakly

viable and some power of each element lies in a subgroup of S if and only if S is a band of semigroups which are ideal extensions of groups by nil semigroups. Also, Theorem 9, the result referred to above, proves, in part, that S is weakly viable, separative and some power of each element lies in a subgroup if and only if S is a band of groups.

The final result of this paper indicates the extent of the class of weakly viable semigroups. In particular, viable semigroups and medial semigroups are weakly viable.

2. **Decomposition theorems.** In order to shorten our proofs we will say $a|b$ if $b \in S^1 a \cap a S^1$. Then the set-valued function $\mathfrak{W}$ on S, defined above, is given by $\mathfrak{W}(a) = \{e | e \in E, a|e\}$.

Define the congruence relation β on S by $a \beta b$ if and only if $\mathfrak{W}(cad) = \mathfrak{W}(cbd)$ for every $c, d \in S^1$.

THEOREM 1. *Let S be weakly viable. Then:*

(i) *β is a congruence relation on S and S/β is a band.*

(ii) *Each β-class contains at most one idempotent and a group ideal whenever it contains an idempotent.*

(iii) *If β' is any other congruence on S satisfying* (i) *and* (ii) *then $\beta' \subseteq \beta$.*

(iv) *$\mathscr{H} \subseteq \beta$. In fact, $\mathscr{H}$ is contained in any congruence β such that S/β is a band.*

(v) *If $e \in \mathfrak{W}(a)$, then $ae \beta e \beta ea$.*

PROOF. (i) Suppose $a \beta b$ and $s \in S$. Let $c, d \in S^1$. Then $\mathfrak{W}(casd) = \mathfrak{W}(ca(sd)) = \mathfrak{W}(cbsd)$. Hence $as \beta bs$. Similarly $sa \beta sb$ and hence β is a congruence. From the definition of weakly viable, $\mathfrak{W}(cad) = \mathfrak{W}(ca^2d)$. Hence $a \beta a^2$ and S/β is a band.

(ii) Suppose $e_1 \beta e_2$. Then $\mathfrak{W}(ce_1d) = \mathfrak{W}(ce_2d)$, for all $c, d \in S^1$. In particular, $\mathfrak{W}(e_1) = \mathfrak{W}(e_2)$. Since $e_i \in \mathfrak{W}(e_i)$, $i = 1, 2$, we have $e_2|e_1$ and $e_1|e_2$. Thus $e_1 \mathscr{H} e_2$ and by [2, Lemma 2.15], $e_1 = e_2$. Hence each β-class contains at most one idempotent. Let B be a β-class containing the idempotent e and let $a \in B$. Then $e \in \mathfrak{W}(e) = \mathfrak{W}(a) = \mathfrak{W}(a^2)$. Hence there exists $x \in S$ such that $a^2x = e$. But $a \beta a^2$ implies $ax \beta a^2x = e$. Hence $ax \in B$ and $a(ax) = e$. Thus e is a right zeroid of B. Similarly e is a left zeroid and hence by [1, §2.5, Exercise 6], B has a group ideal.

(iii) Suppose $a \beta' b$ and that $e \in \mathfrak{W}(cad)$ with $c, d \in S^1$. Then $cad|e$ so there exists $x \in S$, such that $cadx = e$. Hence $cbdx \beta' e$. Since e is in the group ideal of its β'-class we have $cbdx|e$. Hence $cbdxx' = e$, for some $x' \in S$. Consequently $e \in cbdS$. Similarly $e \in Scbd$ and so $\mathfrak{W}(cad) \subseteq \mathfrak{W}(cbd)$. By symmetry, $\mathfrak{W}(cbd) \subseteq \mathfrak{W}(cad)$ and hence $\mathfrak{W}(cad) = \mathfrak{W}(cbd)$. Therefore $a \beta b$.

(iv) Suppose $a \mathscr{H} b$. Then, in particular, there exist $x, y \in S^1$ such that $a = bx$, $b = ya$. Since $b \beta b^2$, $a \beta a^2$ and β is a congruence we have $a = bx \beta b^2x = ba = ya^2 \beta ya = b$. Hence $a \beta b$.

(v) Let $ax=e=ya$. Then $ea=ya^2 \beta ya=e=ax \beta a^2x=ae$. Hence $ea \beta e \beta ae$.

COROLLARY. *Let S be a band of one idempotent semigroups with group ideals. Then this decomposition coincides with the β-decomposition of Theorem 1 and hence is unique.*

PROOF. It is easy to prove that S is weakly viable (see Theorem 6). Let β' be the congruence induced by this decomposition. By Theorem 1(iii), $\beta' \subseteq \beta$. Suppose $a \beta b$. Since each component contains an idempotent there exist e_1, $e_2 \in E$ such that $a \beta' e_1$, $b \beta' e_2$. But since $\beta' \subseteq \beta$, $a \beta e_1$ and $b \beta e_2$. Hence $e_1 \beta e_2$, which implies $e_1=e_2$. Consequently $a \beta' b$.

LEMMA 2. *Let I be an idempotent free ideal of S. Then S is weakly viable if and only if the Rees factor semigroup S/I is weakly viable.*

PROOF. Let 0 denote the equivalence class I in S/I. Since I is idempotent free $E(S/I)=E(S)\cup 0$. Let $e \in E(S)$. Since I is an ideal, $x|e$ in S if and only if $x|e$ in S/I. But for every $a \in S/I$, $a|0$. Hence $\mathfrak{W}(x, E(S))\cup\{0\}= \mathfrak{W}(x, E(S/I))$ for all $x \in S$. The result easily follows.

A semigroup S is *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E$.

LEMMA 3. *The following are equivalent.*
 (i) *For every $a \in S$, $S^1 a S^1 \cap E \neq \varnothing$.*
 (ii) *S is E-inversive.*
 (iii) *For every $a \in S$, $\mathfrak{W}(a) \neq \varnothing$.*

PROOF. (i)$\Rightarrow$(ii). Let $a \in S$. There exist s, $t \in S^1$ such that $sat \in E$. But then $a(tsats) \in E$. Hence S is E-inversive.

(ii)$\Rightarrow$(iii). Let $a \in S$. Then $ax \in E$ for some $x \in S$. Let $s=xax$. Then $e_1=as$ and $e_2=sa$ are idempotents. (This observation is due to Croisot. See [1, §3.2, Exercise 8].) There exists y such that $e_2 e_1 y \in E$. Hence $e_3=e_1 y e_2 e_1 y e_2 \in E$ and $e_1 e_3=e_3=e_3 e_2$. Thus $a(se_3)=e_3=(e_3 s)a$, so that $a|e_3 \in E$. Hence $\mathfrak{W}(a) \neq \varnothing$.

(iii)$\Rightarrow$(i) a fortiori.

If B is a β-class of S, define $\mathfrak{W}(B)=\mathfrak{W}(b)$ where b is any element of B. The definition is independent of the choice of b in B. If $s \in S$, let $[s]$ denote the β-class of s.

LEMMA 4. *Let S be weakly viable and B a β-class of S. If $e \in \mathfrak{W}(B)$, then $B \cup [e]$ is a subsemigroup of S having $[e]$ as an ideal and the group ideal of $[e]$ as a group ideal. Hence if B is idempotent free, $B \cup [e]$ is a one idempotent semigroup with group ideal.*

PROOF. Let $x, y \in B \cup [e]$. If x and y are both in B or both in $[e]$, then clearly $xy \in B \cup [e]$. So for $B \cup [e]$ to be a semigroup it suffices to prove that

yx, $xy \in B \cup [e]$ whenever one is in B and one is in $[e]$. Suppose $x \in B$, $y \in [e]$. Then $e \in \mathfrak{W}(x)$ so by Theorem 1(v), $xe\ \beta\ e\ \beta\ ex$. But $e\ \beta\ y$, so $ex\ \beta\ yx$ and $xe\ \beta\ xy$. Hence xy, $yx \in [e]$. This shows $B \cup [e]$ is a semigroup with $[e]$ as an ideal. Moreover, for every $a \in B \cup [e]$, ae, $ea \in [e]$. Hence ea, $ae | e$ and e is a zeroid. By [**2**, §2.5, Exercise 6], $B \cup [e]$ has a group ideal. It is easy to see that the group ideal of $B \cup [e]$ is in fact the group ideal of $[e]$.

PROPOSITION 5. *S is weakly viable and E-inversive if and only if $S = \bigcup \{S_\alpha | \alpha \in \Omega\}$ where each S_α is a one idempotent semigroup with group ideal and for each $a \in S$, c, $d \in S^1$, there exists S_α such that cad, $ca^2d \in S_\alpha$.*

PROOF. Assume S is weakly viable and E-inversive. Let $\mathfrak{F} = \{B | B \in S/\beta$, B is idempotent free$\}$. By Lemma 3, $\mathfrak{W}(B) \neq \varnothing$, so by the axiom of choice there exists a function $f: \mathfrak{F} \to E$, such that $f(B) \in \mathfrak{W}(B)$. Define

$$S_B = B \qquad \text{if } B \text{ has an idempotent,}$$
$$= B \cup [f(B)] \quad \text{otherwise.}$$

Then clearly $S = \bigcup \{S_B | B \in S/\beta\}$ and each S_B has exactly one idempotent. That each S_B has a group ideal follows from Lemma 4. Now let $a \in S$; c, $d \in S^1$. Then cad, ca^2d belong to the same β-class, say B, and hence cad, $ca^2d \in B \subseteq S_B$.

Conversely, let $e \in E$. If $e \in \mathfrak{W}(cad)$ then there exists $x \in S$ such that $cadx = e$. By hypothesis ca^2dx and $cadx = e$ belong to the same S_α. So e must be the idempotent of the group ideal of S_α. Hence there exists $y \in S_\alpha$ such that $ca^2dxy = e$. Hence $e \in ca^2dS$. Similarly $e \in Sca^2d$. Hence $e \in \mathfrak{W}(ca^2d)$. A similar argument proves that $\mathfrak{W}(ca^2d) \subseteq \mathfrak{W}(cad)$ and we conclude that S is weakly viable. That S is E-inversive is immediate.

REMARK. A proof similar to that above proves that S is viable and E-inversive if and only if $S = \bigcup \{S_\alpha | \alpha \in \Omega\}$ where each S_α is a one idempotent semigroup with group ideal and such that for all a, $b \in S$, there exist S_α such that ab, $ba \in S_\alpha$.

THEOREM 6. *The following are equivalent.*

(i) *S is weakly viable.*

(ii) *S is a band of semigroups having at most one idempotent and a group ideal whenever it has an idempotent.*

(iii) *S is an ideal extension of an idempotent free semigroup (possibly empty) by a weakly viable E-inversive semigroup.*

(iv) *S is an ideal extension of an idempotent free semigroup (possibly empty) by a semigroup T which is the union of one idempotent semigroups with group ideals such that if c, $d \in T^1$, $a \in T$, cad and ca^2d have a common component.*

PROOF. (i)$\Rightarrow$(ii) follows from Theorem 1.

(ii)$\Rightarrow$(i). Let $e \in \mathfrak{W}(cad)$. Then $cadx = e$ for some $x \in S$. Since a and a^2 lie in the same component, $cadx$ and ca^2dx lie in the same component. Hence e must be in the group ideal of this component. So, there exists $y \in S$ such that $ca^2dxy = e$. Hence $e \in ca^2dS$. Similarly $e \in Sca^2d$ and $\mathfrak{W}(cad) \subseteq \mathfrak{W}(ca^2d)$. A similar argument shows $\mathfrak{W}(ca^2d) \subseteq \mathfrak{W}(cad)$ and thus equality holds. Therefore S is weakly viable.

(iii)$\Leftrightarrow$(iv) follows from Proposition 5.

(i)$\Rightarrow$(iii). Let $I = \{a \in S | S^1 a S^1 \cap E = \varnothing\}$. If $I = \varnothing$, S is E-inversive by Lemma 3. Otherwise I is an idempotent free ideal of S. S/I is weakly viable by Lemma 2 and E-inversive since it has a zero.

(iii)$\Rightarrow$(i) follows from Lemma 2 and Proposition 5.

COROLLARY. *Let S be viable. Then S is weakly viable and if δ is defined by $a \, \delta \, b$ if and only if $\mathfrak{W}(a) = \mathfrak{W}(b)$, then $\beta = \delta$.*

PROOF. That S is weakly viable follows from Theorem 6 of [3]. It is clear that $\beta \subseteq \delta$. Theorem 3 of [3] shows that δ satisfies (i) and (ii) of Theorem 1, so by (iii) of Theorem 1, $\delta \subseteq \beta$. Hence $\delta = \beta$.

THEOREM 7. *The following are equivalent.*

 (i) *S is weakly viable and a power of each element lies in a subgroup.*

(ii) *S is a band of semigroups which are ideal extensions of groups by nil semigroups.*

PROOF. (i)$\Rightarrow$(ii). Let B be any β-class and let $b \in B$. By the hypothesis there exists $e \in E$ and a positive integer i such that $b^i \, \mathscr{H} \, e$. By Theorem 1(iv), $b^i \, \beta \, e$. But $b^i \, \beta \, b$ and hence $e \in B$. Thus B has a group ideal K with $e \in K$. Since $b^i \, \mathscr{H} \, e$, $b^i = b^i e \in K$. Hence B/K is nil. Therefore each β-class of S is an ideal extension of a group by a nil semigroup and the result follows from Theorem 1.

(ii)$\Rightarrow$(i) follows from Theorem 6.

COROLLARY. *S is weakly viable and periodic (finite) if and only if S is a (finite) band of semigroups which are ideal extensions of periodic (finite) groups by (finite) nil semigroups.*

A semigroup S is *separative* if $x^2 = xy = y^2$ $(x, y \in S)$ implies $x = y$.

The proof of the following lemma is contained in the proof of Corollary 9 of [3].

LEMMA 8. *Let S be an ideal extension of a group by a nil semigroup. If S is separative then S is a group.*

The equivalence of (ii) and (iii) in the following theorem was obtained by Clifford in [1] (or see [2, §4.2, Exercise 3]).

THEOREM 9. *The following are equivalent.*

(i) *S is weakly viable, separative and a power of each element lies in a subgroup.*

(ii) *S is a band of groups.*

(iii) *For every a, b∈S, abS=a^2bS, Sba=Sba^2 and S is left and right regular.*

(iv) *S is weakly viable, left and right regular.*

(v) *S is weakly viable and regular.*

(vi) *S is weakly viable and a union of groups.*

PROOF. (i)⇒(ii). By Theorem 7, S is a band of semigroups which are ideal extensions of groups by nil semigroups. Since each one of these semigroups is separative, they are groups by Lemma 8.

(ii)⇒(iii) from [1, Theorem 7].

(iii)⇒(iv) follows immediately from the observation that S is weakly viable if and only if $cadS \cap Scad \cap E = ca^2dS \cap Sca^2d \cap E$.

(iv)⇒(v). By [2, Theorem 4.3], left and right regular imply regular.

(v)⇒(vi). This will also follow from [2, Theorem 4.3] once we show that weakly viable and regular imply right regular. Suppose $a \in S$, and $axa=a$. Now $\mathfrak{W}(ax)=\mathfrak{W}(a^2x)$ and $e=ax \in \mathfrak{W}(ax)$. Thus there exists $y \in S$ such that $a^2xy=e$. Hence $a^2xya=ea=a$ and S is right regular.

(vi)⇒(i). A union of groups is always separative.

THEOREM 10. *Suppose a, b, c, d∈S, abcd=e∈E implies that there exist p, q∈S such that acp=e, qbd=e. Then S is weakly viable. Hence every medial (abcd=acbd) semigroup is weakly viable.*

PROOF. Let $a \in S$ and suppose $e \in \mathfrak{W}(cad)$. Then there exist $u, v \in S$ such that $cadu=e=vcad$. Hence $(ca)(duc)(ad)u=e$, so $(ca)(ad)p=e$. Similarly $q(ca)(ad)=e$. Hence $e \in \mathfrak{W}(ca^2d)$. A similar argument shows $\mathfrak{W}(ca^2d) \subseteq \mathfrak{W}(cad)$. Hence S is weakly viable.

It is immediate that medial semigroups satisfy the condition of the theorem.

We conclude with some problems for further study.

1. Is a (weakly viable) E-inversive semigroup a disjoint union of semigroups with group ideals.

2. Characterize semigroups which are disjoint unions of semigroups having at most one idempotent.

3. Theorem 6 of [3] shows that a semilattice of semigroups having at most one idempotent is viable. Is a band of semigroups having at most one idempotent weakly viable?

4. Characterize those semigroups which are rectangular bands of semigroups having at most one idempotent.

5. Study semigroup rings of weakly viable, E-inversive semigroups. In this regard Weissglass has proved the following unpublished result: Let D be a viable, E-inversive semigroup such that the only elements satisfying $x^3 = x^2$ are idempotents. Let R be a ring with no nil ideals. If the period of every element of finite order is prime to the characteristic of R then the semigroup ring of D over R has no nil ideals.

One might be led to the conjecture that a weakly viable E-inversive semigroup is a band of one idempotent semigroups. In fact even a commutative E-inversive semigroup need not be such a band. Let $\langle a \rangle$ be the infinite cyclic semigroup generated by a and let T be the semigroup given by

	0	e_1	e_2
0	0	0	0
e_1	0	e_1	0
e_2	0	0	e_2

Let $S = T \cup \langle a \rangle$ and define $t\langle a \rangle = \langle a \rangle t = t$ for every $t \in T$. Then S is commutative and E-inversive, but it is easy to check that S is not a band of one idempotent subsemigroups.

REFERENCES

1. A. H. Clifford, *Bands of semigroups*, Proc. Amer. Math. Soc. 5 (1954), 499–504. MR 15, 930.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. 1, Math. Surveys, no. 7, Amer. Math. Soc., Providence R.I., 1961. MR 24 #A2627.

3. M. S. Putcha and J. Weissglass, *A semilattice decomposition into semigroups having at most one idempotent*, Pacific J. Math. 38 (1971).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106

Current address (Mohan S. Putcha): Department of Mathematics, University of California, Berkeley, California 94720