PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

ON A CONVERSE OF YOUNG'S INEQUALITY

IH-CHING HSU

ABSTRACT. A converse of Young's inequality is proved through the formulation of a functional inequality.

The converse of Young's inequality presented in this note differs from the converses given in Mitrinović [2] and Takahashi [3].

THEOREM. *Let F denote the collection of all real-valued functions f defined for all* $0 \leqq t < \infty$ *such that*

(i) $f(0)=0$,

(ii) *f is continuous for* $0 \leqq t < \infty$ *(from the right at* $t=0$),

(iii) *f is strictly increasing for* $0 \leqq t < \infty$, *and* $\lim_{t\to\infty} f(t) = +\infty$.

Suppose that $T: F \to F$ is an operator such that

$$T[f](0) = 0, \qquad f \in F, \tag{1}$$

$$xy \leqq T[f](x) + T[f^{-1}](y), \qquad f \in F,\ x \geqq 0, y \geqq 0, \tag{2}$$

$$\forall f \in F, \quad ab = T[f](a) + T[f^{-1}](b), \quad \text{if } b = f(a). \tag{3}$$

Then

$$T[f](x) = \int_0^x f(t)\,dt, \qquad x \geqq 0.$$

PROOF. From (3), it follows that

$$T[f^{-1}](f(x)) = xf(x) - T[f](x) \quad \text{and} \tag{4}$$

$$T[f](f^{-1}(x)) = xf^{-1}(x) - T[f^{-1}](x) \quad \text{since } f(f^{-1}(x)) = x. \tag{5}$$

In (2), replace y by $f(x)$, x by $x+h$, then use (4) to infer that

$$\begin{aligned}(x+h)f(x) &\leqq T[f](x+h) + T[f^{-1}](f(x)) \\ &= T[f](x+h) + xf(x) - T[f](x).\end{aligned}$$

This implies that

$$hf(x) = (x+h)f(x) - xf(x) \leqq T[f](x+h) - T[f](x) \tag{6}$$

holds for $x>0$ and small h.

Received by the editors February 1, 1971.

AMS 1970 *subject classifications*. Primary 26A87; Secondary 26A84.

With $f^{-1}(t) = x + h$, equality (5) leads to

$$
\begin{aligned}
T[f](x+h) - T[f](x) &= T[f][f^{-1}(t)] - T[f](x) \\
&= tf^{-1}(t) - T[f^{-1}](t) - T[f](x) \\
&= (x+h)f(x+h) - T[f^{-1}](t) - T[f](x) \qquad (7)\\
&= hf(x+h) + xf(x+h) - T[f](x) - T[f^{-1}](t) \\
&\leqq hf(x+h) + T[f^{-1}][f(x+h)] - T[f^{-1}](t) \\
&= hf(x+h),
\end{aligned}
$$

since $xf(x+h) - T[f](x) \leqq T[f^{-1}][f(x+h)]$ by (2). Combine (6) and (7) to infer that

$$
f(x) \leqq \frac{T[f](x+h) - T[f](x)}{h} \leqq f(x+h) \qquad (8)
$$

holds for $x \geqq 0$ and for small positive h. Clearly the reverse inequalities hold for $x > 0$ and for small negative h. The continuity of f implies that $dT[f](x)/dx$ exists and equals $f(x)$ for $x \geqq 0$. Thus $T[f](x) = T[f](x) - T[f](0) = \int_0^x f(t)\,dt$. This completes the proof.

Note. The author would like to acknowledge information from a referee who supplied the references to [2] and [3].

REFERENCES

1. J. B. Diaz and F. T. Metcalf, *An analytic proof of Young's inequality*, Amer. Math. Monthly 77 (1970), 603–609. MR 41 #8608.

2. D. S. Mitrinović, *Analytic inequalities*, Springer-Verlag, New York, 1970.

3. T. Takahashi, *Remarks on some inequalities*, Tôhoku Math. J. 36 (1932), 99–106.

DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115